$$\sum_{ij} (-1)^{i+j} m_{ij} 4^{v-i} > 0?$$

More generally, this study is meant to throw some light on the nature of the numbers $m_{ij}$, which may in time lead to solutions of problems such as the 4-color problem.

[1] Presented to the American Mathematical Society, Oct. 25, 1930.

[2] Birkhoff, G. D., *Ann. Math.*, 2, **14**, No. 1 (42–46).

[3] See Whitney, H., *Bull. Am. Math. Soc.*, Abstract No. 36–11–396.

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## *NON-SEPARABLE AND PLANAR GRAPHS*[1]

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1. *Introduction.*—We shall give here an outline of the main results of a research on non-separable and planar graphs. The methods used are entirely of a combinatorial character; the concepts of rank and nullity play a fundamental rôle. The results will be given in detail in a later paper.

A *graph* $G$ is composed of two sets of symbols: *vertices*, $a, b, \ldots, f$, and *arcs*, $\alpha(ab)$ (or simply $ab$), $\beta(ac), \ldots, \delta(ef)$. A *chain* is a set of distinct arcs and vertices, $ab, bc, \ldots, de$. A *suspended chain* is a chain containing at least two arcs, of which no vertices are on other arcs but the first and last vertices, which are each on at least two other arcs. A *circuit* is a set of distinct arcs and vertices, $ab, bc, \ldots, de, ea$. A *k-circuit* is a circuit containing $k$ arcs. A *subgraph* $H$ of $G$ is a graph formed by dropping out arcs from $G$. Let $V, E, P$ be the number of vertices, arcs and connected pieces in $G$. We define the *rank* $R$ and the *nullity* (or cyclomatic number) $N$ by the equations

$$R = V - P,$$
$$N = E - R = E - V + P.$$

2. *Non-Separable Graphs.*—$G$ is called *non-separable* if it is connected, and if there are no two graphs $G_1$ and $G_2$, each containing at least one arc, which form $G$ if a vertex of one is made to coalesce with a vertex of the other. If $G$ is not non-separable, it is separable. $G$ is called *cyclicly connected* if each pair of vertices is contained in a circuit in $G$.

THEOREM 1. *Let $G$ be a graph containing at least two arcs but no 1-circuit. A necessary and sufficient condition that $G$ be non-separable is that $G$ be cyclicly connected.*[2]

Suppose a connected part of $G$ is separable. We then separate it at a vertex into two parts. We continue in this manner until each connected part is non-separable. We say $G$ is decomposed into its non-separable *components*, or its components, simply.

THEOREM 2. *A graph G may be decomposed into its components in a unique manner.*

THEOREM 3. *Let* $R_1, \ldots, R_m, N_1, \ldots, N_m$ *be the ranks and nullities of the components* $G_1, \ldots, G_m$ *of G. Then*

$$R = R_1 + \ldots + R_m,$$

$$N = N_1 + \ldots + N_m.$$

THEOREM 4. *A necessary and sufficient condition that G be non-separable is that there exist no division of its arcs into two groups* $G_1$ *and* $G_2$ *so that*

$$R = R_1 + R_2.$$

We shall say two non-separable graphs form a *circuit of graphs* if they have at least two common vertices. A set of three or more non-separable graphs form a circuit of graphs if they are in cyclic order, each graph has just one vertex in common with the graph following, these vertices are distinct, and no other pair of these graphs have a vertex in common.

THEOREM 5. *Let* $G_1, \ldots, G_m$ *be a set of non-separable graphs, each containing at least one arc, and let G be formed by letting vertices and arcs of different of these graphs coalesce. Then the following four statements are all equivalent:*

(1) $G_1, \ldots, G_m$ *are the non-separable components of G.*

(2) *No two of the graphs* $G_1, \ldots, G_m$ *have an arc in common, and there is no circuit in G containing arcs of more than one of these graphs.*

(3) *No subset of these graphs form a circuit of graphs.*

(4)

$$R = R_1 + \ldots + R_m.$$

THEOREM 6. *A non-separable graph of nullity* 1 *is a circuit.*

THEOREM 7. *If G is a non-separable graph of nullity* $N > 1$, *we can remove an arc or suspended chain, leaving a non-separable graph of nullity* $N-1$.

3. *Homeomorphic Graphs.*—If we can rename the vertices and arcs of a graph $G'$, giving distinct vertices and distinct arcs different names, so that $G'$ becomes identical with $G$, we say $G$ and $G'$ are *homeomorphic*. If $G$ and $G'$ become homeomorphic when they are decomposed into their non-separable components and any isolated vertices are dropped out, we say they are *equivalent*.

THEOREM 8. *Let G and G′ be two graphs containing no isolated vertices or* 2 *circuits. Let there exist a* 1-1 *correspondence between their arcs so that*

(1) *To each* 1-*circuit in one corresponds a* 1-*circuit in the other,*

(2) *To each pair of arcs in one having* $i$ *common vertices corresponds a pair of arcs in the other having* $i$ *common vertices,* $(i = 0, 1)$, *and*

(3) $G$ *and* $G'$ *contain the same number of connected pieces whose arcs are of the form ab, ac, ad.*

*Then* $G$ *and* $G'$ *are homeomorphic.*

4. *Duals, Planar Graphs.*—If $H_1$ is a subgraph of $G$, that subgraph $H_2$ of $G$ containing those arcs not in $H_1$ is called the complement of $H_1$ in $G$. Let $R$, $R'$, $r$, $r'$, etc., stand for the ranks of $G$, $G'$, $H$, $H'$, etc., respectively, with similar definitions for nullity. Suppose there is a 1-1 correspondence between the arcs of $G$ and $G'$ such that if $H$ is any subgraph of $G$ and $H'$ is the complement of the corresponding subgraph of $G$, then

$$r' = R' - n.$$

We say then $G'$ is a *dual* of $G$.

THEOREM 9. *Let* $G'$ *be a dual of* $G$. *Then*

$$R' = N, \; N' = R.$$

THEOREM 10. *If* $G'$ *is a dual of* $G$, $G$ *is a dual of* $G'$.

THEOREM 11. *If* $G'$ *and* $G''$ *are equivalent and* $G'$ *is a dual of* $G$, *then* $G''$ *is a dual of* $G$.

THEOREM 12. *Let* $G_1, \ldots, G_m$ *and* $G_1', \ldots, G_m'$ *be the components of* $G$ *and* $G'$, *respectively, and let* $G_i'$ *be a dual of* $G_i$, $i = 1, \ldots, m$. *Then* $G'$ *is a dual of* $G$.

THEOREM 13. *Let* $G$ *and* $G'$ *be dual graphs, and let* $G_1, \ldots, G_m$ *be the components of* $G$. *Let* $G_1', \ldots, G_m'$ *be the corresponding subgraphs of* $G'$. *Then* $G_1', \ldots, G_m'$ *are the components of* $G'$, *and* $G_i'$ *is a dual of* $G_i$, $i = 1, \ldots, m$.

From this theorem follows the

THEOREM 14. *A dual of a non-separable graph is non-separable.*

Let a topological graph be a graph whose arcs are sets of points in 1-1 correspondence with the unit interval, the end vertices of the arcs corresponding with the ends of the interval. We may associate a topological graph with each graph, or abstract graph, and conversely. A topological graph is called planar if it can be mapped in a 1-1 continuous manner on a plane (or sphere). An abstract graph is planar if the corresponding topological graph is planar.

THEOREM 15. *A necessary and sufficient condition that a graph be planar is that it have a dual.*

[1] Presented to the American Mathematical Society, Oct. 25, 1930.

[2] Compare Whyburn, G. T., these PROCEEDINGS, 13, 1927 (31–38), Theorem 1; Ayres, W. L., *Am. J. Math.*, 51, 1929(577–594), Theorem 10.