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The First Algorithm for Linear Programming: An Analysis of Kantorovich's Method

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Abstract

An analysis is given of Kantorovich's method of resolving multipliers. It is shown that the method is equivalent to a parametric method but that it is also equivalent with the simplex method with a special rule for the choice of the new basic variable.

1. Introduction

In 1939 L. V. Kantorovich published a paper 'Mathematical Methods of Organizing and Planning Production' of which an English translation appeared in *Management Science* in 1960 (see reference 5). The problems considered in this paper were linear programming problems of a somewhat special type, mainly because Kantorovich had some specific applications in mind. The method which Kantorovich proposed for solving these problems and which he called 'The Method of Resolving Multipliers' was not explicitly described, though a number of examples of applications were given.

In a note which precedes the *Management Science* translation Koopmans makes the following comments (see reference 9):

> The computational procedure, as described in Appendices 1 and 2, invites further research. At first sight it does not seem equivalent to Dantzig's simplex method, although it is in a broader category with it in that it is also an iterative procedure in which trial vectors of quantities and of prices are successively revised in the light of profitability criteria. It is desirable that the performance characteristics of a completely specified procedure based on the author's indications be studied in relation to the classes of matrices considered in the paper.

In spite of the widespread recognition which Kantorovich's work has enjoyed, his method has not been analyzed in the light of the currently available theory and methods of linear programming. It is the purpose of this paper to do this now. Our attention will be mainly focussed on the 1939 article.

2. Kantorovich's Production Planning Problems

Kantorovich indicates three types of problems which are of increasing complexity and which originate from production planning problems. In the following we shall use a somewhat different notation from Kantorovich's.

Problem A deals with the allocation of machines to products. Suppose there are m (possibly different) machines which can produce any of n products; if machine i is used for product j, it can produce per time unit a_{ij} units of product j. Let x_{ij} be the number of time units of machine i allocated to product j and let z_j be the total number of units of product j

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produced. We then have the following equation for a feasible allocation of machines to products per time unit:

$$z_j = \sum_i a_{ij} x_{ij}, \qquad j = 1, \ldots, n, \tag{1}$$

$$\sum_j x_{ij} \leq 1, \qquad i = 1, \ldots, m, \tag{2}$$

$$x_{ij} \geq 0, \qquad i = 1, \ldots, m, j = 1, \ldots, n. \tag{3}$$

The objective function is more complicated than in the Western linear programming literature. First, let us assume that the products are all parts of the same article, that is, each finished article requires exactly one unit of each product. If z_o is the number of finished articles which should be maximized, we have $z = \min_j z_j$ or

$$z_o \leq z_j \qquad j = 1, \ldots, n, \tag{4}$$

and the objective function is

$$\text{Max } f = z_o \tag{5}$$

Obviously, the problem (1) – (5) is a linear programming problem, though one of a rather special type. A rather trivial generalisation is obtained if the number of machines of type i is not 1, but b_i, in which case the conditions (2) become

$$\sum_i x_{ij} \leq b_i, \qquad i = 1, \ldots, m. \tag{6}$$

A further generalization is obtained if the number of parts of type j required for the article is not 1, but a_j; conditions (4) then become

$$a_j z_o \leq z_j, \qquad j = 1, \ldots, n. \tag{7}$$

It is well known that with each constraint of a linear programming problem is associated a dual variable. Kantorovich calls these dual variables 'resolving multipliers' (see reference 5) and 'objectively determined evaluations' in a later work (see reference 7). The latter term reflects the fact that they arise from the problem itself and not from prices or costs given from the outside. Since the problem as stated above contains no cost or price-elements, this term is appropriate in this situation. However, it is debatable whether many situations exist in which there are no alternative uses for any of the production factors.

The problem (5), (7), (1), (6), (3) can also be interpreted in a different way. Instead of interpreting z_o as the quantity produced of an article of which the z_j are the quantities of its parts, we may assume that the products $j = 1, \ldots, n$ should be produced in fixed proportions, $a_1, a_2, \ldots, a_n$ and that z_o indicates the overall fulfilment of the plan which should be maximized.

This implies a preference function of the type indicated by the lines a_1 and a_2 in Figure 1, in which the two products should always be produced in a ratio 2 to 1 and no substitution is possible between the two products. If (5) and (7) are replaced by the objective function

$$f = \sum_i a_j z_j, \tag{8}$$

the direction of the objective function is the same, but now substitution between products is possible at the ratio 1 to 2 (see the lines b_1 and b_2 in Figure 1). The traditional indifference curves of elementary economic theory can be considered as intermediate cases (see curves c_1 and c_2). In market economics with free competition we would have an objective function of the type given by (8).

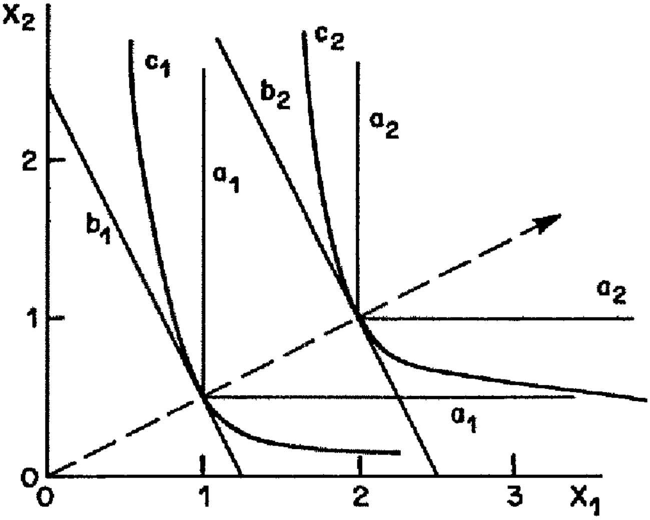

Figure 1.

Kantorovich (see reference 5) mentions an extension of problem A in which there are a number of articles produced from parts, each article having a given value in terms of money; the total value of production can then be maximized. This, of course, corresponds with an objective function of the type (8), but Kantorovich does not use this form in the remainder of his work.

Problem A is related to a problem called the generalized transportation problem or weighted distribution problem (see Dantzig, reference 2). In this case, both inputs and outputs are prescribed:

$$\sum_j x_{ij} \leq b_i, \qquad i = 1, \ldots, m, \tag{9}$$

$$\sum_i a_{ij} x_{ij} \geq a_j \qquad j = 1, \ldots, n, \tag{10}$$

but costs should be minimized:

Minimize

$$f = \sum_{i,j} c_{ij} x_{ij}. \tag{11}$$

Kantorovich (see reference 5) dealt with costs in what was probably prescribed manner in the Soviet Union by imposing maximum quantities on each cost category, such as electricity, labour, water, etc. If c is the available amount of a resource and c_{ij} is the amount of this resource used if product j is produced on machine i, then the following constraint should be satisfied:

$$\sum_{i,j} c_{ijk} x_{ij} \leq c. \tag{12}$$

Problem A with an additional constraint of the type (12) constitute what Kantorovich calls problem B.

Kantorovich also formulates an extension of problem A, which he calls problem C. In problem A, each machine can be used to produce a particular part or product. Instead of this, it is assumed that each machine can be used for different methods of production which may have as outputs a number of products. In other terms, for each machine there are a number of activities which have possibly a number of outputs. If $a_{ij\ell}$ is the output of part or product j when machine i is used in the 1-th method of production, equation (1) should be replaced by

$$z_j = \Sigma_i \Sigma_\ell a_{ij\ell} x_{i\ell}, \tag{13}$$

and (2) is replaced by

$$\Sigma_i x_{i\ell} \leq 1, \qquad \ell = 1, \ldots L,$$

or

$$\Sigma_\ell x_{i\ell} \leq b_i, \qquad \ell = 1, \ldots L. \tag{14}$$

These constraints, combined with (5), (7), and (12) together form an activity analysis model of considerable generality. Compared with the usual activity analysis model with given prices, the following differences stand out: (a) the treatment of the objective function as indicated above; (b) the treatment of costs as indicated in problem B; (c) the fact that there are only inputs and outputs and no intermediate goods.

3. A Parametric Method and the Simplex Method

In what follows we shall show that Kantorovich's method is essentially equivalent to the simplex method. This will be done using his example of calculations for a numerically specified case of Problem A. First it will be shown that the problem can be solved by means of a parametric method. This method is then shown to be equivalent to the simplex method. After that, the parametric method is shown to be equivalent to another parametric method. In the next section, this second parametric method and Kantorovich's method are shown to be equivalent.

The following formulation of Problem A will be used: Maximize

$$f = z_o$$

subject to

$$\begin{aligned}
&a_{11}x_{11} + a_{21}x_{21} + \ldots + a_{m1}x_{m1} \geq a_1 z_o,\\
&a_{12}x_{12} + a_{22}x_{22} + \ldots + a_{m2}x_{m2} \geq a_2 z_o,\\
&\ldots\\
&a_{1n}x_{1n} + a_{2n}x_{2n} + \ldots + a_{mn}x_{mn} \geq a_n z_o,\\
&x_{11} + x_{12} + \ldots + x_{1n} \leq b_1,\\
&x_{21} + x_{22} + \ldots + x_{2n} \leq b_2,\\
&\ldots\\
&x_{m1} + x_{m2} + \ldots + x_{mn} \leq b_m,\\
&x_{11}, \ldots, x_{mn} \geq 0.
\end{aligned}$$

In terms of vectors and matrices, the problem may be written as: Maximize

$$f = z_o \tag{15}$$

subject to

$$-Ax + az_o \leq 0, \tag{16}$$

$$Bx \leq b, \tag{17}$$

$$x \geq 0, \tag{18}$$

where

$$a = \begin{bmatrix} a_1 \\ a_2 \\ . \\ . \\ a_n \end{bmatrix}, \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ b_m \end{bmatrix},$$

$$A = \begin{bmatrix} a_{11} & 0 & . & 0 & a_{21} & 0 & . & 0 & .. & a_{m1} & 0 & . & 0 \\ 0 & a_{12} & . & 0 & 0 & a_{22} & . & 0 & .. & 0 & a_{m2} & . & 0 \\ . & . & . & . & . & . & . & . & .. & . & . & . & . \\ 0 & 0 & . & a_{1n} & 0 & 0 & . & a_{2n} & .. & 0 & 0 & . & a_{mn} \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 1 & . & 1 & 0 & 0 & . & 0 & . & . & 0 & 0 & . & 0 \\ 0 & 0 & . & . & 1 & 1 & . & 1 & . & . & 0 & 0 & . & 0 \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & . & 0 & 0 & 0 & . & 0 & . & . & 1 & 1 & . & 1 \end{bmatrix}.$$

Consider now replacing the objective function (15) by

$$f = \bar{\lambda}'Ax + (\mu - a'\bar{\lambda})\, z_o, \tag{19}$$

where $\bar{\lambda}$ is a given nonnegative vector and μ is a variable parameter. It is obvious that for $\mu \to \infty$ the solutions to both problems are the same; furthermore, for $\bar{\lambda} = 0$, the problems are identical apart from the scalar factor μ.

Let us now consider Kantorovich's example for Problem A in which the machines are three different excavators which can move earth of three different types. The productivity of each excavator for each type of soil is given by the following array:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{matrix} = \\ = \\ = \end{matrix} \begin{bmatrix} 105 & 56 & 56 \\ 107 & 66 & 83 \\ 64 & 38 & 53 \end{bmatrix}.$$

There is one excavator of each type and the three soil types should be moved in the same proportions. This implies

$$a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

The values of the elements of $\bar{\lambda}$, in Kantorovich's terms, the initial values of the resolving multipliers, are

$$\bar{\lambda}_1 = 3.62, \bar{\lambda}_2 = 6.25, \bar{\lambda}_3 = 5.208.$$

The x-variables will be indicated as x_{ij} which represents the number of hours of machine i allocated to product j. The objective function is to maximize z_o which is in this case the fraction of the work to be done in one hour, the work consisting of equal amounts of the three different soil types.

TABLE 1. Tableaux for Parametric and Simplex Method

Tableau	B.V.	V.B.V.	z_o	x_{11}	x_{12}	x_{13}	x_{21}	x_{22}	x_{23}	x_{31}	x_{32}	x_{33}
	f_c	0	15.078	−380.43	−350	−291.67	−387.68	−412.5	−432.29	−231.88	−237.5	−276.04
	f_μ	0	−1									
	z_1	0	1	−105			−107			−64		
0	z_2	0	1		−56			−66			−38	
	z_3	0	1			−56			−83			−53
	y_1	1		1	1	1						
	y_2	1					1	1	1			
	y_3	1								1	1	1
			z_o	y_1	x_{12}	x_{13}	x_{21}	x_{22}	y_2	x_{31}	x_{32}	y_3
	f_c	1088.76	15.078	380.43	30.43	88.76	44.61	19.79	432.29	44.16	38.54	276.04
	f_μ	0	−1									
	z_1	105	1	105	105	105	−107			−64		
1	z_2	0	1		−56			−66			−36	
	z_3	136	1			−56	83	83	83	53	53	53
	x_{11}	1		1	1	1						
	x_{23}	1					1	1	1			
	x_{33}	1								1	1	1
			z_2	y_1	x_{12}	x_{13}	x_{21}	x_{22}	y_2	x_{31}	x_{32}	y_3

Table 1. *Continued.*

Tableau	B.V.	V.B.V.	z_o	x_{11}	x_{12}	x_{13}	x_{21}	x_{22}	x_{23}	x_{31}	x_{32}	x_{33}
	f_c	1088.76	−15.078	380.43	874.995	88.76	44.61	1015.17	432.29	44.16	611.637	276.04
	$f_\mu z_o$	0	1		−56			−66			−38	
	z_1	105	−1	105	161	105	−107	66		−64	38	
2	z_3	136	−1		56	−56	83	*149*	83	53	91	53
	x_{11}	1		1	1	1						
	x_{23}	1					1	1	1			
	x_{33}	1								1	1	1
			z_2	y_1	x_{12}	x_{13}	x_{21}	z_3	y_2	x_{31}	x_{32}	y_3
	f_c	162.1619	−8.2678	380.43	493.45116	470.3001	−520.887	−6.8132	−133.2074	−316.9107	−8.3661	−85.0607
	$f_\mu z_o$	60.258	0.557		−31.184	−24.816	36.762	0.443	36.762	23.496	2.326	23.496
	z_1	44.742	−0.557	105	*136.184*	129.816	−143.762	−0.443	−36.762	−87.496	−2.326	−23.496
3	x_{22}	0.913	−0.00671		0.376	−0.376	0.557	0.00671	0.557	0.356	0.611	0.356
	x_{11}	1		1	1	1						
	x_{23}	0.087	0.00671		−0.376	0.376	−0.557	−0.00671	0.443	−0.356	−0.611	−0.356
	x_{33}	1								1	1	1
			z_2	y_1	z_1	x_{13}	x_{21}	z_3	y_2	x_{31}	x_{32}	y_3
	f_c	0	−6.25		−3.6234			−5.208				
	$f_\mu z_o$	70.503	0.429	24.045	0.229	4.9119	3.8405	0.342	28.343	3.459	1.793	18.115
	x_{12}	0.3285										
4	x_{22}	0.789										
	x_{11}	0.6715										
	x_{23}	0.211										
	x_{33}	1										

If the objective function (19) is used, Tableau 0 of Table 1 is the set-up tableau for the problem. The basic solution of this tableau is nonoptimal, but if x_{11}, x_{23}, and x_{33} are made basic as indicated, an optimal solution results for $\mu = 0$ (see Tableau 1). This solution is optimal for $\mu \leq 15.078$. By using standard parametric programming for a parametric objective function, z_o is made basic and z_2 leaves the basis, resulting in Tableau 2, which is optimal for $15.078 \leq \mu \leq 15.381$. Note that the rows of f_μ and z_o are identical. x_{22} then enters the basis and z_3 leaves the basis, resulting in Tableau 3, which is optimal for $15.381 \leq \mu \leq 15.824$. Now x_{12} enters the basis, z_1 leaves the basis and Tableau 4 results. The basic solution of this tableau is optimal for $\mu \geq 15.824$, so that it must give the optimal solution to the original problem with $f = z_o$.

If λ_1, λ_2, and λ_3 had been zero, the entire f_c -row becomes zero in Tableau 0. For the parametric method the same choice of pivots could have been made, though any variable with a negative element in the f_μ-row could have been made basic.

Alternatively, we could have interpreted the sequence of pivot choices as an application of the simplex method, with the f_μ, z_o- row as objective function row. In the simplex method any variable with a negative element in the objective function row can be made basic. In the parametric method the selection of the new basic variable also depended on negative elements in the f_μ-row, but the elements in the f_c-row which are dependent on the $\overline{\lambda}$'s made the choice among the non-basic variables with a negative element in the f_μ-row determinate. Hence, the parametric method can be interpreted as equivalent to the simplex method with a special choice of the new basic variable.

To show the equivalence between Kantorovich's method and the parametric method, it is useful to consider the parametric problem in a somewhat different form. Instead of the problem: Maximize

$$f = \overline{\lambda}'Ax + (\mu - a'\overline{\lambda})z_o$$

subject to

$$\begin{aligned} -Ax + az_o &\leq 0, \\ Bx &\leq b, \\ x &\geq 0 \end{aligned}$$

for $0 \leq \mu \leq \infty$, we may consider the problem: Minimize

$$f = \overline{\lambda}'Ax - \overline{\lambda}'az_o$$

subject to

$$\begin{aligned} -Ax &\leq -az_o, \\ Bx &\leq b, \\ x &\geq 0, \end{aligned}$$

where z_o is the variable parameter, which is varied from 0 to ∞.

Both problems are said to be parametrically equivalent, which means that a variation of μ in the first problem and a variation of z_o in the second problem lead to the same sequence of solutions for critical values of both parameters. For details about parametric equivalence, see van de Panne (reference 10).

Table 2 gives the tableau for a parametric variation of z_o in the second problem. Tableau 0 gives the set-up problem; the initial solution is generated in Table 1, which gives an optimal solution for $z_o = 0$. Then z_o is varied upwards. After two steps it is found that z_o cannot be increased any further, so that the maximum value which z_o can take is

307.77/4.366 = 70.5. The complete solution can easily be found by pivoting on the element 4.366; this solution is given in Tableau 4.

To facilitate comparison with Kantorovich's method, the corresponding dual problem and its solutions are given in Table 3. The dual variables of the constraints referring to the products are indicated as λ_1, λ_2, and λ_3; these have been given initial values $\bar{\lambda}_1$, $\bar{\lambda}_2$, and $\bar{\lambda}_3$. The nonbasic variables in this dual problem are deviations from these initial values and are therefore indicated by λ_1^*, λ_2^*, and λ_3^*; as soon as they become basic, their full value is used and the stars are deleted.

4. Kantorovich's Method and the Parametric Method

First, a short description of Kantorovich's method is given. The starting point is the generation of an initial feasible and optimal solution for the given initial values of soil moved, which are $\bar{\lambda}_1 = 3.62$, $\bar{\lambda}_2 = 6.25$, $\bar{\lambda}_3 = 5.208$. The value of one hour of the first excavator used for the three types of soil is then

380.43, 350, 219.67,

and for the second and third excavator

387.68, 412.5, 432.29,
231.88, 237.5, 276.04.

The excavators are then allocated to the type of soil which gives the maximum value, so that the first excavator is allocated to soil 1 and the other two to soil 3. This results in the following amounts of soils 1, 2, and 3 produced:

105, 0, 136.

Since we want to maximize $z_0 = \text{Min. } z_j^*$, where z_j^* is the amount of soil j produced, we want to increase z_2^* and in order to do this optimally, we increase the value of soil 2, indicated by λ_2, upwards from $\bar{\lambda}_2 = 6.25$ until the value of the production of soil 2 by one of the machines becomes equal to the value of its present production so that it can be switched from its present use to soil 2. The best use of machine 1 is at present soil 1 where it yields 380.43. The value of λ_2 for which it is profitable to use machine 1 for soil 2 is 380.43/56 = 6.79 and the corresponding values for machines 2 and 3 are 432.29/66 = 6.55 and 276.04/38 = 7.26. Hence λ_2 is increased to 6.55; at this point machine 2 can be used both for soil 2 and 3. The use of machine 2 for soil 2, and together with this $z_0 = \min z_j^*$, can be increased until (1) the machine 2 is entirely used for soil 2 and not for soil 3 at all; (2) the production of soil 2 has become equal to that of soil 1 ($z_1^* = z_2^*$), (3) the production of soil 2 has become equal to that of soil 3 ($z_2^* = z_3^*$). In this case (3) happens first. Hence a new solution is obtained from the equations

$$\begin{aligned}
&x_{11} = 1, \\
&x_{22} + x_{23} = 1, \\
&x_{33} = 1, \\
&z_1^* = 105\, x_{11}, \\
&x_2^* = 66\, x_{22}, \\
&z_3^* = 83\, x_{23} + 53\, x_{33}.
\end{aligned} \tag{20}$$

The solution of this system of equations is

$$\begin{aligned}
&x_{11} = 1, x_{22} = .913, x_{23} = .087, x_{33} = 1, \\
&z_1^* = 105, z_0 = z_2^* = z_3^* = 60.26.
\end{aligned}$$

TABLE 2. Tableaux for Equivalent Parametric Problem

Tableau	B.V.	c-t	z_o-t	x_{11}	x_{12}	x_{13}	x_{21}	x_{22}	x_{23}	x_{31}	x_{32}	x_{33}
	f		−15.078	−380.43	−350	−291.67	−387.68	−412.5	−432.29	−231.88	−237.5	−276.04
	z_1		−1	−105			−107			−64		
	z_2		−1		−56			−66			−38	
0	z_3		−1			−56			−83			−53
	y_1	1		1	1	1						
	y_2	1					1	1	1			
	y_3	1								1	1	1
		c-t	z_o-t	y_1	x_{12}	x_{13}	x_{21}	x_{22}	y_2	x_{31}	x_{32}	y_3
	f	1088.76	−15.078	380.43	30.43	88.76	44.61	19.79	432.29	44.16	38.54	276.04
	z_1	105	−1	105	105	105	−107			−64		
	z_2	0	−1		−56			−66			−38	
1	z_3	136	−1			−56	83	83	83	53	53	53
	x_{11}	1		1	1	1						
	x_{23}	1					1	1	1			
	x_{33}	1								1	1	1
		c-t	z_o-t	y_1	x_{12}	x_{13}	x_{21}	z_2	y_3	x_{31}	x_{32}	y_3

TABLE 2. *Continued.*

Tableau	B.V.	c-t	z_o-t	x_{11}	x_{12}	x_{13}	x_{21}	x_{22}	x_{23}	x_{31}	x_{32}	x_{33}
	f	1088.76	−15.378	380.43	13.638	88.76	44.76	0.2998	432.29	44.66	27.146	276.04
	z_1	105	−1	105	105	105	−107			−64		
	x_{22}	0	0.01515		0.848			−0.01515			0.576	0
2	z_3	136	−2.2575		*−70.424*	−56	83	1.2576	83	53	5.212	53
	x_{11}	1		1	1	1						
	x_{23}	1	−0.01515		−0.848		1	0.01515	1		−0.576	
	x_{33}	1								1	1	1
				y_1	z_3	x_{13}	x_{21}	z_2	y_2	x_{31}	x_{32}	y_3
	f	1115.097	−15.815	380.43	0.437	77.915	60.684	*0.543*	448.36	54.424	28.155	266.304
	z_1	307.77	*−4.366*	105	1.491	21.506	16.75	1.875	123.75	15.022	7.771	79.021
	x_{22}	−1.6386	−0.01205		0.01205	−0.6747	1		1	0.63855	−0.63855	0.63855
3	x_{12}	−1.9312	0.03206		−0.0142	0.79518	−1.1786	0.01786	−1.1786	−0.75258	−0.0741	−0.75258
	x_{11}	2.9311	−0.03206	1	0.0142	0.2048	1.1786	0.1786	1.1786	0.75258	0.07401	0.75258
	x_{23}	−6.3855	−0.01205		−0.01205	6.7470		5.5511		−6.3855	−6.3855	−6.3855
	x_{33}	1								1	1	1
			z_1	y_1	z_3	x_{13}	x_{21}	z_2	y_2	x_{31}	x_{32}	y_3
	z_o	70.493	0.229	24.05	0.3415	4.9258	3.865	0.429	28.344	3.4405	1.7799	18.099
	x_{22}	0.7892										
4	x_{12}	0.3286										
	x_{11}	0.6714										
	x_{23}	0.2108										
	x_{33}	1										

TABLE 3. Solution of Dual Problem

Tableau	B.V.	V.B.V.	λ_1^*	λ_2^*	λ_3^*	u_1	u_2	u_3
	g_c					−1	−1	−1
	g_{zo}	−15.078	1	1	1			
	v_{11}	−380.43	105			−1		
	v_{12}	−350		56		−1		
0	v_{13}	−291.67			56	−1		
	v_{21}	−387.68	107				−1	
	v_{22}	−412.5		66			−1	
	v_{23}	−432.29			83		−1	
	v_{31}	−231.88	64					−1
	v_{32}	−237.5		38				−1
	v_{33}	−276.04			53			−1
			λ_1^*	λ_2^*	λ_3^*	v_{11}	v_{23}	v_{33}
	g_c	1088.76	−105		−136	−1	−1	−1
	g_{zo}	−15.078	1	1	1			
	u_1	380.43	−105			−1		
	v_{12}	30.43	−105	56		−1		
	v_{13}	88.76	−105		56	−1		
1	v_{21}	44.61	107		−83		−1	
	v_{22}	19.79		66	−83		−1	
	u_2	432.29			−83		−1	
	v_{31}	44.16	64		−53			−1
	v_{32}	38.54		38	−53			−1
	u_3	276.04			−53			−1
			λ_1^*	v_{22}	λ_3^*	v_{11}	v_{23}	v_{33}
	g_c	1088.76	−105		−136	−1	−1	−1
	g_{zo}	−15.378	1	−0.01515	2.2576		0.01515	
	u_1	380.43	−105			−1		
	v_{12}	13.638	−105	−0.848	70.424	−1	0.848	
	v_{13}	88.76	−105		56	−1		
2	v_{21}	44.76	107		−83		−1	
	λ_2	6.55		0.01515	−1.2576		−0.01515	
	u_2	432.29			−83		−1	
	v_{31}	44.16	64		−53		−1	
	v_{32}	27.146		−576	−5.212	0.576	−1	
	u_3	276.04			−53		−1	
			λ_1^*	v_{22}	v_{12}	v_{11}	v_{22}	v_{33}
	g_c	1115.098	−307.77	−1.6386	1.9312	−2.93115	6.3855	−1
	g_{zo}	−15.815	4.366	0.01205	−0.03206	0.03206	−1.2048	
	u_1	380.43	−105			−1		
	λ_3	5.402	−1.491	−0.01205	0.0142	−0.0142	0.01205	
	v_{13}	77.915	−21.506	0.6747	−0.79518	−0.2048	−6.7470	
3	v_{21}	60.684	−16.75	−1	1.1786	−1.1786		
	λ_2	6.793	−1.875		0.01786	−0.01786	−5.5511	
	u_2	448.36	−123.75	−1	1.1786	−1.1786		

TABLE 3. *Continued.*

Tableau	B.V.	V.B.V.	λ_1^*	λ_2^*	λ_3^*	u_1	u_2	u_3
	v_{31}	54.424	−15.022	−0.63855	0.75258	−0.75258	6.3855	−1
	v_{32}	28.155	−7.771	−0.63855	0.0741	−0.07401	6.3855	−1
	u_2	286.304	−79.021	−6.3855	0.75258	−0.75258	6.3855	−1
				v_{22}	v_{12}	v_{11}	v_{23}	v_{33}
	g		70.493	−0.7892	−0.3286	−0.6714	−0.2108	−1
	λ_1		0.229					
	u_1		24.05					
	λ_3		0.3415					
	v_{13}		4.9258					
4	v_{21}		3.8365					
	λ_2		0.429					
	u_2		28.344					
	v_{31}		3.4405					
	v_{32}		1.7799					
	u_3		18.099					

At this point z_o can be increased by increasing both z_2^* and z_3^*, which is done by increasing both λ_2 and λ_3, but λ_2 is linked.to λ_3 via machine 2. λ_3 should be increased until it becomes profitable to allocate another machine, in this case machine 1, to soil 2 or 3. (For reasons of computional convenience Kantorovich chose to decrease λ_1 instead.)

We then solve for λ_3 as follows: if machine 1 is to be used for soil 2, we have:

$$56\lambda_2 = 380.43,$$
$$66\lambda_2 = 83\lambda_3,$$

which implies $\lambda_3 = 5.4$; if machine 1 is to be used for soil 3, we have:

$$56\lambda_3 = 380.43$$

or $\lambda_3 = 6.79$. Other allocations are irrelevant, so that we take $\lambda_3 = 5.4$, at which point x_{12} can become positive.

The value of x_{12} is determined by the equations of (20), except that the first equation is now

$$x_{11} + x_{12} = 1.$$

Since there is an additional variable, there should be an additional equation. This equation can be $z_1^* = z_2^* = z_o$, which leads to

$$z_o = 70.5, x_{11} = .6715, x_{12} = .3285, x_{22} = .789, x_{23} = .211, x_{33} = 1.$$

Other possibilities are $x_{11} = 0$ and $x_{22} = 0$ which both lead to lower z_o values or infeasible solutions.

This new solution must be optimal because it gives for the resolving multipliers $\lambda_1 = 3.62$, $\lambda_2 = 6.72$, $\lambda_3 = 5.4$, an allocation in which each machine is allocated to its best use.

To point out the equivalence of this method with the parametric method applied in Tables 2 and 3, let us first consider the dual problem, which is:
Minimize

$$f = u_1 + u_2 + u_3 - z_0(\lambda_1 + \lambda_2 + \lambda_3)$$

subject to

$$\begin{aligned}
105\lambda_1 - u_1 + v_{11} &= 0,\\
56\lambda_2 - u_1 + v_{12} &= 0,\\
56\lambda_3 - u_1 + v_{13} &= 0,\\
107\lambda_1 - u_1 + v_{21} &= 0,\\
66\lambda_2 - u_2 + v_{22} &= 0,\\
83\lambda_3 - u_2 + v_{23} &= 0,\\
64\lambda_1 - u_3 + v_{31} &= 0,\\
38\lambda_2 - u_3 + v_{32} &= 0,\\
53\lambda_2 - u_3 + v_{33} &= 0.\\
\lambda_1, \lambda_2, \lambda_3, u_1, u_2, u_3, v_{11}, \ldots, v_{33} &\geq 0.
\end{aligned}$$

The u-variables stand for the dual variables or shadow prices of the machines, which Kantorovich does not explicitly introduce, and the v-variables for minus the profitability of using machine i for soil j; these profitabilities should be nonpositive for optimal solutions. In Kantorovich's method, the v-variables, if nonzero, are implied by the difference of the maximum value which a machine can produce and the value in a particular allocation:

$$\begin{aligned}
237.5 + 38\lambda_2^* + v_{32} &= u_3,\\
276.04 + 53\lambda_3^* + v_{33} &= u_3.
\end{aligned}$$

These equations correspond with the rows of Tableau 0 in Table 3.

If $\lambda_1^* = \lambda_2^* = \lambda_3^* = 0$, $u_1 = 380.43$, $v_{11} = 0$, then v_{12} and v_{13} will be nonnegative. This corresponds with pivoting in the element -1 in the row of v_{11} and the column of u_1. The other two pivots are explained in the same way. Hence, taking the allocation of the machines with the maximum value results in a feasible solution to the dual problem and therefore an optimal solution to the primal problem. From Tableau 0 of Table 2 it is obvious that this solution will also be feasible for the primal problem.

Tableau 1 of Table 2 then indicates that an increase in z_0 from 0 is stopped at $z_0 = 0$ because otherwise the slack variable z_2 becomes negative. This is equivalent with min $z_0 = \Sigma_j = z_i^* = z_2^*$. In order to increase z_0, z_2^* should be increased, and to do this optimally, we should increase λ_2. In Tableau 1 of Table 3 such an increase in λ_2^* is considered and it is found that for $\lambda_2^* = 19.79/66 = .30$, v_{22} becomes 0, which is equivalent to $\lambda_2 = 6.55$. If $v_{22} = 0$, x_{22} can be increased, which is what happens in the primal problem, Tableau 1 of Table 2. After transformation with -66 as a pivot, Tableau 2 of Table 2 is found.

Now z_0 can be increased from 0; its maximum happens to be 60.26 at which point z_3 becomes 0, which means that $z_0 = z_2^* = z_3^*$; the possibilities connected with $z_1 = 0$ and $x_{11} = 0$ are also considered but they involve negative values for z_3.

Kantorovich then considers an increase in λ_3 to increase $z_0 = z_3^*$. In the tableaux the same thing happens by selecting a pivot in the row of z_3 in Tableau 2 of Table 2 or in the column of λ_3^* in Tableau 2 of Table 3. We shall not go into further details.

The final values of the resolving multipliers are only relative values. The proper values should sum to 1 because of the equation in dual problem:

$$\lambda_1 + \lambda_2 + \lambda \geq 1.$$

Hence the values are $\lambda_1 = .229$, $\lambda_2 = .429$, $\lambda_3 = .342$.

5. Concluding Remarks

Kantorovich gives three more examples of application of his method of resolving multipliers, one for a larger problem of type A and one each for problems of the type B and C (see reference 5). None of these examples follows the described method exactly. There may be two reasons for this, the first one being that Kantorovich attempted to use computational shortcuts and the second that at that time Kantorovich did not find it useful to adjust both primal and dual variables as systematically as can be done. The result is that for these examples his method appears to be a trial-and-error procedure; this may be one of the reasons why so little has been written about the method.

In his book *The Best Use of Economic Resources* which was published in Russia in 1959, Kantorovich restates his method, which he then called 'the method of adjusting multipliers (valuations)'. In this restatement the computation of the multipliers or dual variables and of the primal variables is given in a form which is closer to the first equivalent parametric formulation given above, as implemented in Table 1. The form in which the method is stated is similar to that in which the primal-dual method of Dantzig, Ford and Fulkerson (see reference 3) is stated; Kantorovich indicates: 'In very recent years a similar method has begun to be used in other countries'.

In this respect it can be noted that the primal-dual method can also be formulated as a parametric method.[1] However, the situations in which both methods are used are different. Whereas the primal-dual method starts with an infeasible but optimal solution, the method of resolving multipliers starts with a feasible solution but an incorrect objective function, which is gradually changed into the correct one.

The most striking difference between Kantorovich's method and modern linear programming methods is that Kantorovich always returns to the original primal and dual equation systems, while the modern methods work with transformed forms of only one of these systems. In principle, Kantorovich has to solve an equation system for each possible new basic variable and for each possible leaving basic variable, but knowledge of the structure of the equation systems facilitates this to a large extent. This may be one of the reasons why Kantorovich deals with linear programming problems of a given structure.

Some early methods for quadratic programming, for instance Houthakker's capacity method (reference 4) and Theil and van de Panne's combinatorial method (reference 12) at each step start from the original equation system and are rather inefficient for that reason. It is not surprising that for linear programming a method with the same characteristics has been proposed. The efficiency of Dantzig's simplex method compared with Kantorovich's method is obvious.

On the other hand, the pathbreaking nature of Kantorovich's work in 1939 is beyond dispute and not until one or two decades later did western literature begin to display insights in linear programming models equal to those given by Kantorovich's paper.

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[1]See Kelley (reference 8), van de Panne and Whinston (reference 11) or van de Panne (reference 10).

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