U. S. AIR FORCE

PROJECT RAND

RESEARCH MEMORANDUM

NOTES ON LINEAR PROGRAMMING---PART XXXI:
A PRIMAL--DUAL ALGORITHM

G. B. Dantzig
L. R. Ford, Jr.
D. R. Fulkerson

RM-1709

9 May 1956

Assigned to ______________________

This research is sponsored by the United States Air Force under contract No. AF 49(638)-700 monitored by the Directorate of Development Planning, Deputy Chief of Staff, Research and Technology, Hq USAF.

The RAND *Corporation*
1700 MAIN ST. • SANTA MONICA • CALIFORNIA

SUMMARY

The procedure developed by two of the authors (Ford and Fulkerson) for solving transportation problems is a natural extension of the Kuhn-Egervary method for solving assignment problems. In the present paper the procedure is extended further to the general linear programming case.

CONTENTS

A PRIMAL-DUAL ALGORITHM

1. INTRODUCTION

Kuhn, basing his investigation on the work of Egervary, has developed a special routine for solving assignment problems [10]. Paul Dwyer has proposed a similar type of approach for the more general transportation problem [7]. Also, along the same lines, two of the present authors [8] have developed, in connection with maximal-flow problems in networks, a special algorithm that has been extended to Hitchcock-Koopmans transportation problems [3,9].

Experiments indicate that this technique is very efficient. Our purpose is to generalize the process to solve the general linear programming problem. As stated here, it becomes a special variant of the simplex process [4,2,6,5], that promises to reduce the number of iterations by doing away with the two-phase process. (In Phase I a basic feasible solution is determined; this is needed to initiate Phase II, in which an optimal basic feasible solution is obtained.)

Any feasible solution to the dual system may be used to initiate the proposed method. Associated with the dual solution is a "restricted" primal problem that requires optimization. When the solution of the restricted primal problem has been accomplished, an improved solution to the dual system can be obtained. This in turn gives rise to a new restricted problem to be optimized. After a finite number of improvements of the dual, an optimal solution is obtained for both the primal and dual systems.

What distinguishes the transportation case from the more general case is that for the former the optimization of the primal auxiliary problem can be accomplished without the use of the simplex process, whereas the generalization appears to require this process. Thus it might seem that we are proposing a new algorithm to replace the ordinary simplex algorithm when, in fact, the simplex algorithm itself is imbedded within the proposed algorithm.

Actually, the entire process, as we view it, may be considered to be a way of starting with an infeasible basic solution to a linear programming problem and using a feasible solution to the dual if available (otherwise a solution of the modified dual discussed below) to decrease the infeasibility of the primal in such a manner that when a feasible basic solution is obtained, it will be optimal.

2. THE PRIMAL AND DUAL PROBLEMS

We take the primal problem in the following form: Determine values of $x_1,\ldots,x_n$, $\bar{z}$ which minimize $\bar{z}$ subject to

$$(1)\qquad c_1x_1 + c_2x_2 + \ldots + c_nx_n = \bar{z},$$

$$(2)\qquad \begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \qquad (b_1 \geq 0), \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \,, \end{array}$$

$$(3)\qquad x_j \geq 0 \qquad (j=1,\ldots,n),$$

where a_{ij}, c_j, b_i are given constants. There is no loss of generality in assuming $b_i \geq 0$ since the signs of all terms in an equation can be changed if necessary.

The dual problem is to find $\pi_1, \ldots, \pi_m$, $\underline{z}$ which maximize $\underline{z}$ subject to

$$(4) \qquad b_1\pi_1 + b_2\pi_2 + \cdots + b_m\pi_m = \underline{z} ,$$

$$(5) \qquad \begin{array}{l} a_{11}\pi_1 + a_{21}\pi_2 + \cdots + a_{m1}\pi_m \leq c_1 , \\ \vdots \\ a_{1n}\pi_1 + a_{2n}\pi_2 + \cdots + a_{mn}\pi_m \leq c_n . \end{array}$$

It is easy to show that always $\underline{z} \leq \bar{z}$ for any solutions to (1), (2), (3) and to (4), (5). The fundamental duality theorem states that if solutions to the primal exist and $\bar{z}$ has a finite lower bound, then optimal solutions for both primal and dual exist; moreover, any optimal solutions for the primal and dual systems have the property that $\underline{z} = \bar{z}$. Our purpose is to construct a pair of such solutions.

We shall need a feasible solution to the dual to start the algorithm. In many problems, such a solution is readily available. For example, if all $c_j \geq 0$ then obviously $\pi_i = 0$ satisfy (5). In general, however, a dual solution is not available. To get around this, we use a device due to Beale [1] and others, and append to system (2) the relation

$$(6) \qquad x_0 + x_1 + \cdots + x_n = b_0 ,$$

where b_0 is unspecified but is thought of as being arbitrarily large. (More precisely, for each cycle k of the algorithm a value b_0^k can be specified such that any $b_0 \geq b_0^k$ will do. Since there will be only a finite number of cycles, we may take $b_0 \geq \max_k b_0^k$.)

The problem (1), (2), (3), (6) will be called the modified primal. The modified dual corresponding to it is the problem of determining $\pi_0, \pi_1, \ldots, \pi_m$, $\underline{y}$ which maximize $\underline{y}$ subject to

$$(7) \qquad b_0\pi_0 + b_1\pi_1 + \cdots + b_m\pi_m = \underline{y} ,$$

$$(8) \qquad \begin{aligned} \pi_0 & \leq 0 , \\ \pi_0 + a_{11}\pi_1 + \cdots + a_{m1}\pi_m & \leq c_1 , \\ & \vdots \\ \pi_0 + a_{1n}\pi_1 + \cdots + a_{mn}\pi_m & \leq c_n . \end{aligned}$$

Notice that a feasible solution is now readily available; indeed, the set of values $\pi_0 = \min (0, c_1, \ldots, c_n)$, $\pi_i = 0$ for $i > 0$, solves (8).

3. THE EXTENDED PRIMAL PROBLEM

We next consider an extended primal problem with nonnegative error (artificial) variables $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_m$, where the objective (as in Phase I of the simplex process) is to minimize the sum of the errors. Thus we are to determine values of $x_0, x_1, \ldots, x_n$, $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_m$, $\overline{w}$ which minimize $\overline{w}$ subject to

$$(9)\qquad \varepsilon_0 + \varepsilon_1 + \dots + \varepsilon_m = \overline{w},$$

$$(10)\qquad \begin{aligned} x_0 + x_1 + x_2 + \dots + x_n + \varepsilon_0 &= b_0, \\ a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n + \varepsilon_1 &= b_1, \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n + \varepsilon_m &= b_m, \end{aligned}$$

$$(11)\qquad x_j \geq 0,\ \varepsilon_i \geq 0 \qquad (j=0, \dots, n;\ i=0, \dots, m).$$

Theorem 1. Any solutions to (7), (8) and to (9), (10), (11), with the properties that $\overline{w} = 0$, $\pi_0 = 0$, and $x_j = 0$ if the j^{th} relation of (8) is strict inequality, are optimal solutions to the original primal and dual problems.

Proof. Since $\overline{w} = \varepsilon_0 + \dots + \varepsilon_m = 0$ and $\varepsilon_i \geq 0$, we have all $\varepsilon_i = 0$. Thus, because $\pi_0 = 0$, the values of $x_1, \dots, x_n;\ \pi_1, \dots, \pi_m$ satisfy the original primal and dual constraints.

Multiplying the i^{th} relation of (10) by π_i and summing gives

$$(12)\qquad x_1 \sum_{i=1}^{m} a_{i1}\pi_i + x_2 \sum_{i=1}^{m} a_{i2}\pi_i + \dots + x_n \sum_{i=1}^{m} a_{im}\pi_i = \sum_{i=1}^{m} b_i\pi_i .$$

By assumption, all terms in (12) corresponding to $\sum_{i=1}^{m} a_{ij}\pi_i < c_j$ have $x_j = 0$; hence, for all j, the j^{th} term is the same as c_jx_j; i.e., (12) reduces to

$$(13)\qquad \sum_{j=1}^{n} c_jx_j = \sum_{i=1}^{m} b_i\pi_i, \qquad \text{or } \overline{z} = \underline{z},$$

thus establishing that $x_1, \dots, x_n$ and $\pi_1, \dots, \pi_m$ are optimal solutions to the original primal and dual.

4. THE ALGORITHM

Let $\pi_0, \pi_1, \ldots, \pi_m$ be any set of numbers satisfying (8). We associate with this selected solution of the modified dual a restricted primal problem, which is identical with the extended primal problem (9), (10), (11) except that certain x_j variables are "dropped" from the equations. To be more precise, the restricted primal is the extended primal under the added conditions that

$$(14) \qquad x_j = 0 \qquad \text{if } \delta_j < 0,$$

where

$$(14.1) \qquad \delta_0 = \pi_0 ,$$
$$\delta_j = (\pi_0 + \sum a_{ij} \pi_i) - c_j \qquad (j=1, 2, \ldots, n).$$

From (8) it will be noted that $\delta_j \leq 0$ for all j. Thus if we denote the set of indices for which $\delta_j = 0$ by J, the restricted primal is obtained from the extended primal by dropping all variables x_j whose indices j do not belong to J.

The restricted problem is next solved using the revised simplex method [6]. (Since b_0 is unspecified, the values of the variables x_j, ε_i will depend linearly on b_0, with the property that for all b_0 sufficiently large, the solution is feasible.) For example, one could use $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_m$ as an initial set of basic variables and minimize $\bar{w}$ under the assumption (14). (On succeeding restricted primal problems, the prior minimal solution may be taken as a starting solution, as we shall see.) The revised simplex algorithm provides, at the minimum of the restricted problem, optimal solutions x_j, ε_i to the restricted problem, and σ_i to its dual, such that

(15) $$\sigma_i = 1 \text{ if } \varepsilon_i > 0, \ \sigma_i \leq 1 \text{ if } \varepsilon_i = 0,$$
$$\rho_j = \sigma_0 + \sum_{i=1}^{m} a_{ij}\sigma_i = 0 \text{ if } x_j > 0,$$
$$\rho_j \leq 0 \text{ if } x_j = 0 \text{ and } j \varepsilon J.$$

(Notice that ρ_j is defined for <u>all</u> $j = 0,\ldots,n$.)

It follows that the values

(16) $$\pi_i^* = \pi_i + \theta\sigma_i \qquad (i=0,\ldots,m)$$

satisfy the modified dual system (8) for some range of values $0 < \theta \leq \theta_0$. To see this, denote the new values of δ_j by δ_j^*, so that

(17) $$\delta_j^* = \delta_j + \theta\rho_j \,.$$

Now for all $j \varepsilon J$, $\delta_j = 0$ and $\rho_j \leq 0$. Thus, for $j \varepsilon J$, $\delta_j^* \leq 0$ for any $\theta \geq 0$. For $j \not\varepsilon J$, $\delta_j < 0$; hence $\delta_j^* \leq 0$ for $0 < \theta \leq \theta_0$, where

(18) $$\theta_0 = \min_{\rho_j > 0} -\frac{\delta_j}{\rho_j} \ \underline{\text{or}} \ \theta_0 = \infty$$

according as there are or are not any $\rho_j > 0$.

Denoting the new value of $\underline{y}$ by $\underline{y}^*$, we have, from (7) and (16),

(19) $$\underline{y}^* = \sum_{i=0}^{m} b_i\pi_i^* = \sum_{i=0}^{m} b_i(\pi_i + \theta\sigma_i) = \underline{y} + \theta \sum_{i=0}^{m} b_i\sigma_i \,.$$

Now by multiplying the i^{th} equation of (10) by σ_i, summing and noting (15), we get

$$\sum_{i=0}^{m} b_i \sigma_i = \sum_{j=0}^{n} \rho_j x_j + \sum_{i=0}^{m} \varepsilon_i \sigma_i = \sum_{i=0}^{m} \varepsilon_i ;$$

hence by (15), (19) becomes

$$(20) \qquad \underline{y}^* = \underline{y} + \theta \sum_{i=0}^{m} \varepsilon_i = \underline{y} + \theta \bar{w} .$$

We may therefore state the following result.

Theorem 2. An optimal solution to the restricted primal with $\bar{w} > 0$ provides a new feasible solution to the modified dual with a strict increase in the maximizing form $\underline{y}$.

If $\bar{w} > 0$ and all $\rho_j \leq 0$ (so that $\theta_0 = \infty$), then $\bar{w}$ is minimal in the extended primal (9), (10), (11). Hence in this case there is no solution to the original primal system, and the computation terminates.

Assuming that $\bar{w} > 0$ and some $\rho_j > 0$, we repeat the procedure using the new modified dual solution $\pi_i^* = \pi_i + \theta_0 \sigma_i$ and its associated restricted primal. Notice, as was asserted earlier, that we may take the prior minimizing solution for $\bar{w}$ as an initial solution in the new restricted primal, since those j for which $x_j > 0$ have both $\delta_j = 0$ and $\rho_j = 0$, hence $\delta_j^* = 0$.

Theorem 3. The algorithm terminates in a finite number of steps in one of the following situations:

(I) At some stage $\theta_0 = \infty$; hence there is no feasible solution to the original primal.

(IIa) At some stage, $\bar{w} = 0$ and $\pi_0 = 0$; then $x_1, \ldots, x_n$; $\pi_1, \ldots, \pi_m$ are optimal solutions to the original primal and dual.

(IIb) <u>At some stage</u> $\bar{w} = 0$ <u>and</u> $\pi_0 < 0$; <u>then the original primal form</u> $\bar{z}$ <u>has no lower bound</u>.

<u>Proof</u>. We suppose that degeneracy is avoided in each restricted primal by using, if necessary, a perturbation of the b_i. This means that when we minimize $\bar{w}$, the set of $x_j > 0$ and $\varepsilon_i > 0$ constitutes a basic feasible solution. If $\theta_0 = \infty$ at some stage, the computation ends (as we have seen) with the conclusion that no feasible solution to the primal exists. If not, then at each stage there is a $\rho_j > 0$. It follows that x_j may be introduced in place of one of the basic variables, and the nondegeneracy assumption means that $\bar{w}$ will be strictly decreased. Hence the solution of each new restricted primal results in one or more new basic solutions to the extended primal, each with a decrease in $\bar{w}$. Thus no basis can be repeated and the process must terminate in a finite number of steps with a basic solution for which $\bar{w} = 0$.

If $\pi_0 = 0$ when termination occurs, then (see Theorem 1) $x_1, \ldots, x_n$; $\pi_1, \ldots, \pi_m$ are optimal for the original primal and dual. If, on the other hand, $\pi_0 < 0$, then, since $\underline{y} = \bar{z}$, we see from (7) that there is no lower bound for $\bar{z}$ as $b_0 \rightarrow +\infty$.

5. NUMERICAL EXAMPLES

The following examples correspond to the cases of Theorem 3. All variables are nonnegative.

<u>Example I</u>. Consider the equations

$$
\begin{aligned}
-x_1 + x_2 - x_3 \qquad &= 1, \\
x_1 - x_2 \qquad - x_4 &= 1, \\
-x_1 \qquad\qquad &= \bar{z}.
\end{aligned}
$$

The problem is illustrated geometrically in Figure 1.

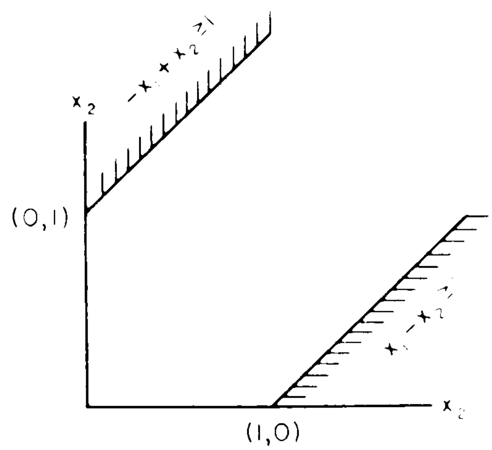

Figure 1
Geometrical Picture for Example I

The extended primal is

$$
\begin{array}{rl}
x_0 + x_1 + x_2 + x_3 + x_4 + \varepsilon_0 & = b_0, \\
-x_1 + x_2 - x_3 \qquad + \varepsilon_1 & = 1, \\
x_1 - x_2 \qquad - x_4 \qquad + \varepsilon_2 & = 1, \\
\varepsilon_0 + \varepsilon_1 + \varepsilon_2 & = \overline{w} .
\end{array}
$$

To start out, take $\pi_0 = -1$, $\pi_1 = \pi_2 = 0$. Then $\delta_0 = -1$, $\delta_1 = 0$, $\delta_2 = -1$, $\delta_3 = -1$, $\delta_4 = -1$. The solution of the corresponding restricted primal is $x_1 = 1$, $\varepsilon_0 = b_0 - 1$, $\varepsilon_1 = 2$, with multipliers $\sigma_0 = 1$, $\sigma_1 = 1$, $\sigma_2 = 0$. Thus $\rho_0 = 1$, $\rho_1 = 0$, $\rho_2 = 2$, $\rho_3 = 0$, $\rho_4 = 1$, and consequently $\theta_0 = {}^1/_2$. Hence

$\pi_0^* = -1/2$, $\pi_1^* = 1/2$, $\pi_2^* = 0$, $\delta_0^* = -1/2$, $\delta_1^* = 0$, $\delta_2^* = 0$, $\delta_3^* = -1$, $\delta_4^* = -1/2$. Using the previous minimal solution $x_1 = 1$, $\epsilon_0 = b_0 - 1$, $\epsilon_1 = 2$ as a starting point and minimizing $\overline{w}$ for the new restricted primal gives the basic solution $x_1^* = (b_0+1)/2$, $x_2^* = (b_0-1)/2$, $\epsilon_1^* = 2$, with multipliers $\sigma_0^* = 0$, $\sigma_1^* = 1$, $\sigma_2^* = 1$. Then $\rho_0^* = 0$, $\rho_1^* = 0$, $\rho_2^* = 0$, $\rho_3^* = -1$, $\rho_4^* = -1$, and $\theta_0^* = \infty$. Since all $\rho_i^* \leq 0$, $\overline{w} = 2$ is minimal and no feasible solution exists to Example I.

Examples IIa, IIb. For the next two examples, we record in Tables 1 and 2 merely the extended primal in detached-coefficient form together with a record of successive values π, δ, σ, ρ and optimizing x, ϵ in the various restricted primals. The problem may be pictured geometrically as in Figure 2, where IIa and IIb denote the normals to the minimizing forms in the direction of decreasing $\overline{z}$.

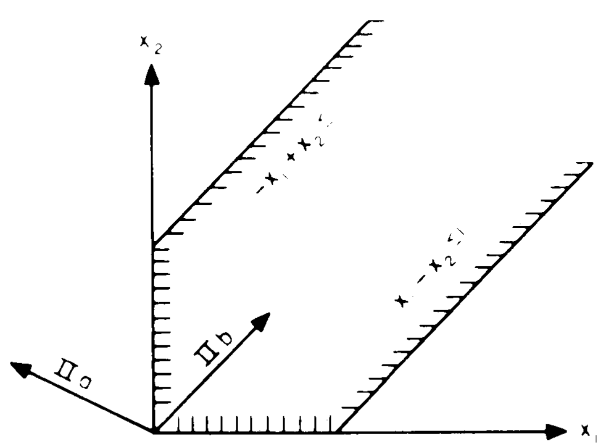

Figure 2
Geometrical Picture for Examples IIa, IIb

Table 1

SOLUTION OF EXAMPLE IIa

	x_0	x_1	x_2	x_3	x_4	ε_0	ε_1	ε_2		π_1	σ_1	π_1	σ_1	π_1	σ_1
	1	1	1	1	1	1	0	0	$= b_0$	−1	1	$-\frac{1}{2}$	1	0	0
	0	−1	1	1	0	0	1	0	= 1	0	0	0	−2	−1	0
	0	1	−1	0	1	0	0	1	= 1	0	1	$\frac{1}{2}$	−1	0	0
	0	2	−1	0	0				$= \bar{z}$	Iteration (0)		(1)		(2)	
						1	1	1	$= \bar{w}$						
δ_j	−1	−3	0	−1	−1				Iteration						
x_j, ε_1	0	0	1	0	0	b_0-1	0	2	(0)						
ρ_j	1	2	0	1	2										
δ_j	$-\frac{1}{2}$	−2	0	$-\frac{1}{2}$	0										
x_j, ε_1	0	0	1	0	2	b_0-3	0	0	(1)						
ρ_j	1	2	0	−1	0										
δ_j	0	−1	0	−1	0										
x_j, ε_1	b_0-3	0	1	0	2	0	0	0	(2)						
ρ_j	0	0	0	0	0										

Table 2

SOLUTION OF EXAMPLE IIb

	x_0	x_1	x_2	x_3	x_4	ε_0	ε_1	ε_2		π_1	σ_1	π_1	σ_1
	1	1	1	1	1	1	0	0	$= b_0$	−1	0	−1	0
	0	−1	1	1	0	0	1	0	= 1	0	1	1	0
	0	1	−1	0	1	0	0	1	= 1	0	1	1	0
	0	−1	−1	0	0				$= \bar{z}$	Iteration (0)		Iteration (1)	
						1	1	1	$= \bar{w}$				
δ_j	−1	0	0	−1	−1				Iteration (0)*				
x_j, ε_1	0	$\frac{1}{2}(b_0-1)$	$\frac{1}{2}(b_0+1)$	0	0	0	0	2					
ρ_j	0	0	0	1	1								
δ_j	−1	0	0	0	0				(1)				
x_j, ε_1	0	$\frac{1}{2}(b_0-3)$	$\frac{1}{2}(b_0-1)$	0	2	0	0	0					
ρ_j	0	0	0	0	0								

*Iteration (0) of IIb requires at least two cycles of the simplex algorithm since basic variables $\varepsilon_0, \varepsilon_1$ are replaced by x_1, x_2.

REFERENCES

[1] Beale, E. M. L., "An Alternative Method for Linear Programming," Proceedings of the Cambridge Philosophical Society, Vol. 50, Part 4, 1954, pp. 513-523.

[2] Charnes, A., W. W. Cooper, and A. Henderson, An Introduction to Linear Programming, John Wiley and Sons, Inc., New York, 1953.

[3] Dantzig, George B., "Application of the Simplex Method to a Transportation Problem," Activity Analysis of Production and Allocation, Cowles Commission for Research in Economics, Monograph 13, 1951, Chapter 23, pp. 359-373.

[4] Dantzig, George B., "Maximization of a Linear Function of Variables Subject to Linear Inequalities," Activity Analysis of Production and Allocation, Cowles Commission for Research in Economics, Monograph 13, 1951, Chapter 21, pp. 339-347.

[5] Dantzig, George B., Notes on Linear Programming—Part XI, Composite Simplex Dual Simplex Algorithm, The RAND Corporation, Paper P-516, April 26, 1954.

[6] Dantzig, George B., Alex Orden, and Philip Wolfe, "Method for Minimizing a Linear Form Under Linear Inequality Restraints," Pacific Journal of Mathematics, Vol. 5, No. 2, June, 1955, pp. 183-195.

[7] Dwyer, Paul S., The Solution of the Hitchcock Transportation Problem with the Method of Reduced Matrices, Engineering Research Institute Report, University of Michigan, 1955.

[8] Fulkerson, D. R., and L. R. Ford, A Simple Algorithm for Finding Maximal Network Flows and an Application to the Hitchcock Problem, The RAND Corporation, Paper P-743, September 26, 1955.

[9] Koopmans, T. C., and Stanley Reiter, "A Model of Transportation," Activity Analysis of Production and Allocation, Cowles Commission for Research in Economics, Monograph 13, 1951, Chapter 14, pp. 222-259.

[10] Kuhn, H. W., "The Hungarian Method for the Assignment Problem," Naval Research Logistics Quarterly, March-June, 1955, pp. 83-98.

[11] Lempke, C. E., "The Dual Method of Solving the Linear Programming Problem," Naval Research Logistics Quarterly, Vol. 1, No. 1, March, 1954, pp. 36-47.