Dating Problem via Linear Programming

Overview

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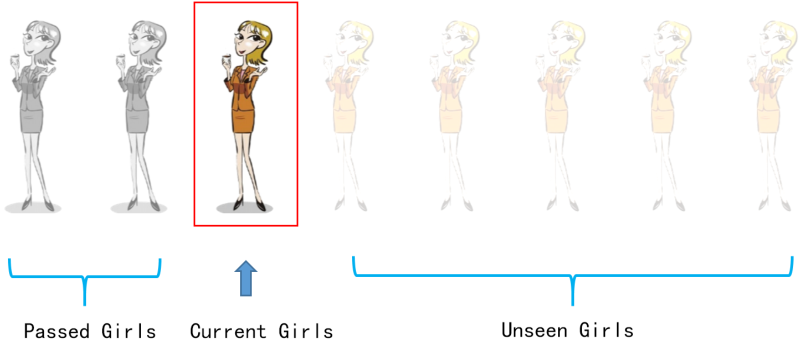
- Certainly, you want to select the best boy/girl as your soul mate
- So, how to design a strategy to maximum the probability that the best one is chosen?

Example



Problem Analysis

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Hardness of Dating Problem

- Lack of information: when you make decision for current person, you have no information about future ones
- One-shot decision: Once you make a decision on a person, you have no chance to regret

Linear Programming Model

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- That's to say

$$x_i = \Pr[\text{The } i\text{-th one is selected} \mid \text{The } i\text{-th one is the best one from 1 to } i] \quad (1)$$

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Remark

It should be emphasis that the probability x_i is defined on the randomness of coming sequence

Linear Programming Model

Analysis

- Clearly, we have

$$
\begin{aligned}
x_i &\leq \Pr[\text{No one is selected before position } i] \\
&= 1 - \sum_{l=1}^{i-1} \Pr[\text{The } l\text{-th one is selected}] \\
&= 1 - \sum_{l=1}^{i-1} \frac{1}{l} \cdot x_l
\end{aligned}
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- It is clearly that

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\begin{aligned}
\Pr[Z] &= \sum_{i=1}^{n} \Pr[\text{The } i\text{-th one is the best person and she/he is chosen}] \\
&= \sum_{i=1}^{n} \Pr[\text{The } i\text{-th one is the best person}] \\
&\quad \cdot \Pr[\text{The } i\text{-th one is selected} | \text{The } i\text{-th one is the best person}] \\
&= \sum_{i=1}^{n} \frac{1}{n} \cdot x_i
\end{aligned} \tag{2}
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Linear Programming Model

Primal

- In summary we have the following LP model:

$$\begin{aligned} \textbf{Maximize: } & \Pr[Z] = \sum_{i=1}^{n} x_i \\ \textbf{s.t. } & x_i \leq 1 - \sum_{l=1}^{i-1} \frac{1}{l} \cdot x_l, \quad 1 \leq i \leq n \\ & x_i \geq 0, \quad 1 \leq i \leq n. \end{aligned} \tag{3}$$

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- So simple a LP model!

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Dual

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- We have the following dual LP of the Dating Problem:

$$\begin{aligned} &\textbf{Minimize:} \quad \sum_{i=1}^{n} y_i \\ &\textbf{s.t.} \quad y_i + \frac{1}{i} \sum_{l=i+1}^{n} y_l \geq \frac{1}{n}, \quad 1 \leq i \leq n \\ &\qquad\quad y_i \geq 0, \qquad\qquad\qquad\quad 1 \leq i \leq n. \end{aligned} \tag{4}$$

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- What's the meaning of y_i?

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- It is an open problem. Clever as you may have some idea.

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- Why strictness is important?

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Strictness of the LP Model

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- *Strictness* means any algorithm can be mapped to a feasible solution of the LP and any solution of the LP can be mapped to an algorithm
- Why strictness is important?
- Only when a LP is strict, the optimal solution of the LP can be an optimal solution for the problem

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- We should show that any feasible solution can be mapped to an algorithm for Dating Problem

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- It's over?
- Naive! We should show that $x_i(\pi) = x'_i$, where $x_i(\pi)$ is the probability that i-th one is selected give she/he is the best one up to i in algorithm π
- Think about why we should show this and show it by yourself (May a homework?)

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- It is not over so far

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- In fact, there is an optimal deterministic algorithm for the Dating Problem:
  - For the first $\frac{n}{e}$ persons, you just dating but select nobody
  - For the rest persons, select the first one that better than all previous
- By this algorithm, you can find your soul mate with probability $\frac{1}{e}$

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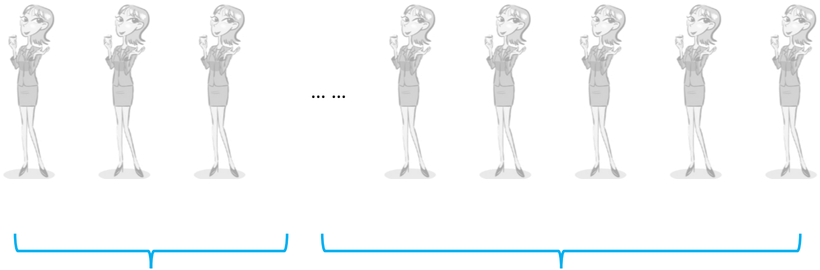

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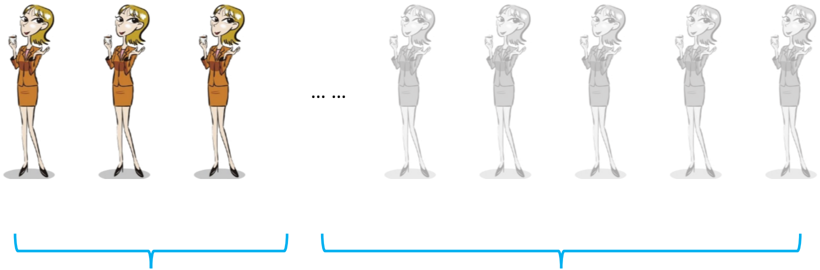

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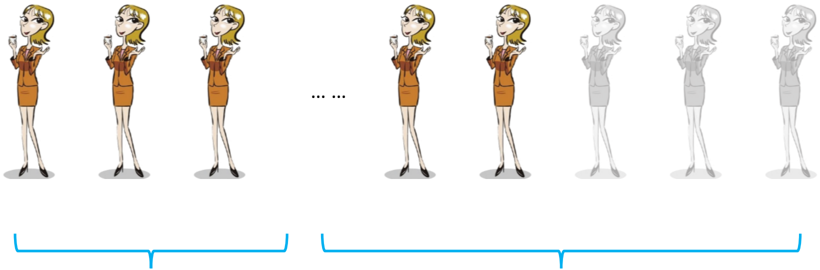

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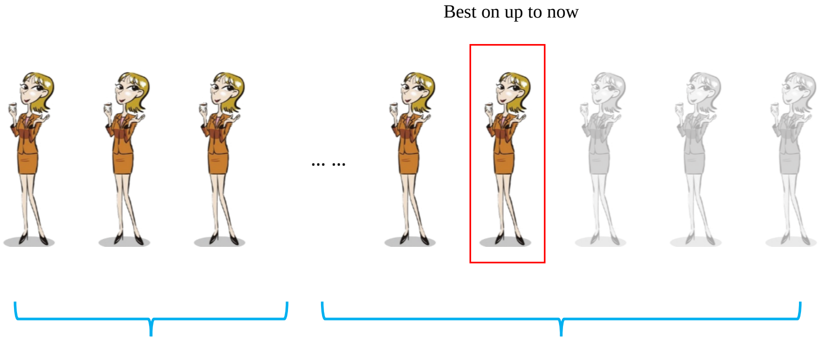

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- By solve some small case using GLPK, we can find the optimal solution is exactly in this pattern
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- We will show the optimality by the Theorem of Complementary Slackness

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Feasible Solution of Dual LP

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- When n is large enough we have $y_i \approx \frac{1-logn/i}{n}$ and it equals to 0 exactly when $i = \frac{n}{e}$!

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- When n is large enough we have $y_i \approx \frac{1 - logn/i}{n}$ and it equals to 0 exactly when $i = \frac{n}{e}$!
- According to the non-negative constraint we set y_i to be 0 when $1 \leq i \leq \frac{n}{e}$

Optimal Algorithm

Feasible Solution of the Primal and Dual LP

- In summary, we have a feasible solution of the dual LP that satisfies:

$$y_i = \begin{cases} 0, & 1 \leq i \leq n/e \\ \dfrac{1}{n} - \dfrac{1}{i} \displaystyle\sum_{l=i+1}^{n} y_l & n/e < i \leq n \end{cases} \tag{5}$$

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- The deterministic algorithm can be mapped to an feasible solution of the primal LP that satisfies:

$$x_i = \begin{cases} 0, & 1 \le i \le n/e \\ 1 - \sum_{l=i+1}^{n} \frac{1}{l} x_l & n/e < i \le n \end{cases} \tag{6}$$

Optimal Algorithm

Slackness Variable of the Primal and Dual LP

- Let xs_i and ys_i stand for the corresponding slackness variable of x_i and y_i, we have

$$xs_i \begin{cases} \geq 0, & 1 \leq i \leq n/e \\ = 0 & n/e < i \leq n \end{cases} \tag{7}$$

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- Easy to check that those tow feasible solutions satisfy the Theorem of Complementary Slackness

Thanks!