# CS711008Z Algorithm Design and Analysis

## Lecture 7. Binary heap, binomial heap, and Fibonacci heap

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# Outline

- Introduction to priority queue
- Various implementations of priority queue:
  - Linked list: a list having $n$ items is too long to support efficient EXTRACTMIN and INSERT operations simultaneously;
  - Binary heap: using a **tree** rather than a **linked list**;
  - Binomial heap: allowing **multiple trees** rather than **a single tree** to support efficient UNION operation
  - Fibonacci heap: implement DECREASEKEY via simply **cutting an edge** rather than **exchanging nodes**, and control a "bushy" tree shape via allowing **at most one child losing** for any node.

Priority queue

# Priority queue: motivation

- Motivation: It is usually a case to **extract the minimum** from a set $S$ of $n$ numbers, **dynamically**.
- Here, the word "dynamically" means that on $S$, we might perform INSERTION, DELETION and DECREASEKEY operations.
- The question is how to organize the data to efficiently support these operations.

# Priority queue

- **Priority queue** is an **abstract data type** similar to stack or queue, but each **element** has a **priority** associated with its **name**.
- A min-oriented priority queue must support the following core operations:
  1. $H$=MakeHeap(): to create a new heap $H$;
  2. Insert$(H, x)$: to insert into $H$ an element $x$ together with its priority
  3. ExtractMin$(H)$: to extract the element with the highest priority;
  4. DecreaseKey$(H, x, k)$: to decrease the priority of element $x$;
  5. Union$(H_1, H_2)$: return a new heap containing all elements of heaps $H_1$ and $H_2$, and destroy the input heaps

# Priority queue is very useful

- Priority queue has extensive applications, such as:
  - Dijkstra's shortest path algorithm
  - Prim's MST algorithm
  - Huffman coding
  - $A^*$ searching algorithm
  - HeapSort
  - ......

## An example: Dijkstra's algorithm

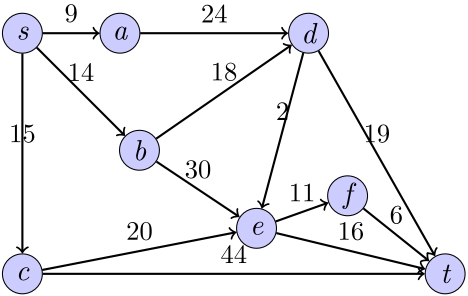
# Dijkstra's algorithm [1959]

DIJKSTRA$(G, s, t)$

$key(s) = 0$; $//key(u)$ stores an upper bound of the shortest distance from $s$ to $u$;
$PQ$. INSERT $(s)$;
**for all** node $v \neq s$ **do**
    $key(v) = +\infty$
    $PQ$. INSERT $(v)$ $//n$ times
**end for**
$S = \{\}$; // Let $S$ be the set of explored nodes;
**while** $S \neq V$ **do**
    $v^* = PQ$. EXTRACTMIN(); $//n$ times
    $S = S \cup \{v^*\}$;
    **for all** $v \notin S$ and $< v^*, v > \in E$ **do**
        **if** $key(v^*) + d(v^*, v) < key(v)$ **then**
            $PQ$.DECREASEKEY$(v, key(v^*) + d(v^*, v))$; $//m$ times
        **end if**
    **end for**
**end while**

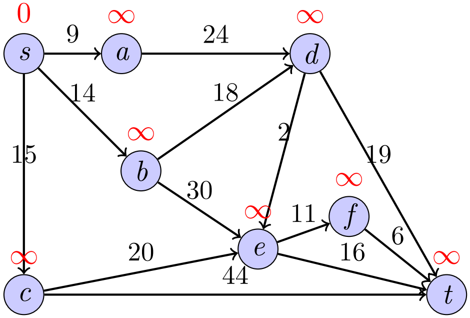
Here $PQ$ denotes a min-priority queue.

# Dijkstra's algorithm: an example



# Initialization

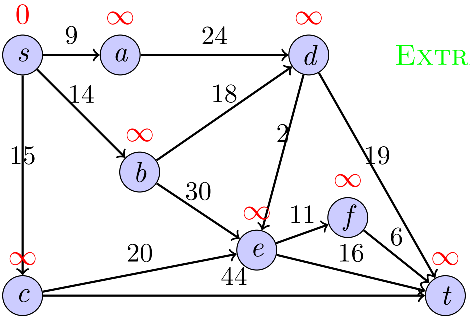
$$S = \{\}$$
$$PQ = \{s(0), a(\infty), b(\infty), c(\infty), d(\infty), e(\infty), f(\infty), t(\infty)\}$$



# Step 1: ExtractMin

$$S = \{\}$$

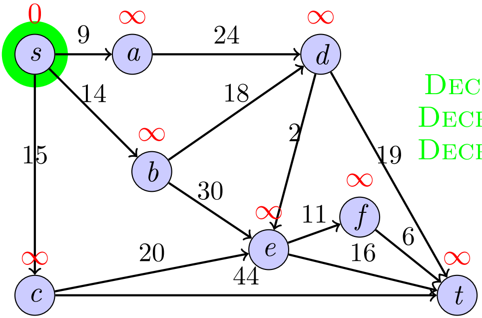
$$PQ = \{s(0), a(\infty), b(\infty), c(\infty), d(\infty), e(\infty), f(\infty), t(\infty)\}$$




ExtractMin returns $s$

# Step 1: DecreaseKey

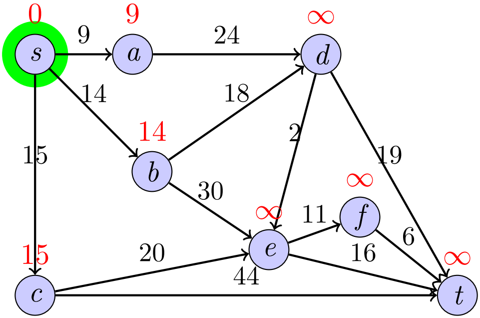
$$S = \{s\}$$
$$PQ = \{a(\infty), b(\infty), c(\infty), d(\infty), e(\infty), f(\infty), t(\infty)\}$$




DecreaseKey$(a, 9)$
DecreaseKey$(b, 14)$
DecreaseKey$(c, 15)$

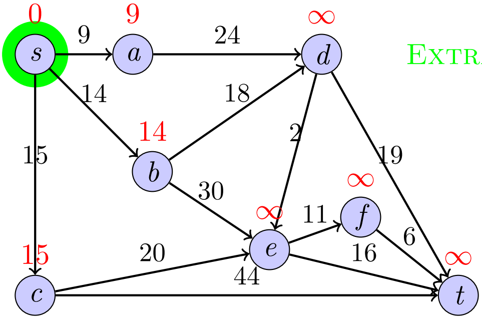
# Step 2: ExtractMin

$$S = \{s\}$$
$$PQ = \{a(9), b(14), c(15), d(\infty), e(\infty), f(\infty), t(\infty)\}$$



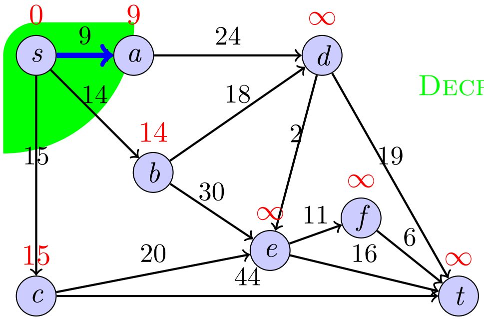
# Step 2: ExtractMin

$$S = \{s\}$$
$$PQ = \{a(9), b(14), c(15), d(\infty), e(\infty), f(\infty), t(\infty)\}$$




ExtractMin returns $a$

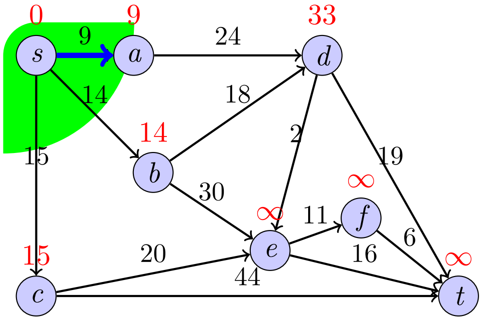
$$S = \{s, a\}$$
$$PQ = \{b(14), c(15), d(\infty), e(\infty), f(\infty), t(\infty)\}$$




DECREASEKEY$(d, 33)$

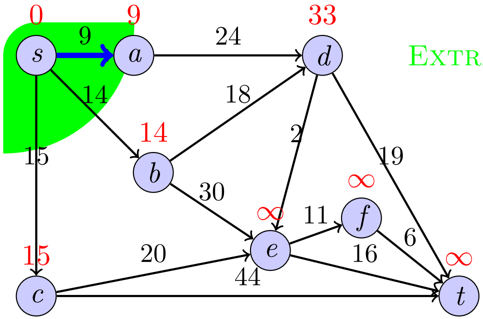
## Step 3: ExtractMin

$$S = \{s, a\}$$
$$PQ = \{b(14), c(15), d(33), e(\infty), f(\infty), t(\infty)\}$$



# Step 3: ExtractMin

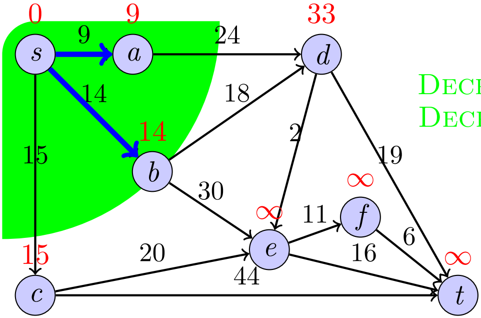
$$S = \{s, a\}$$
$$PQ = \{b(14), c(15), d(33), e(\infty), f(\infty), t(\infty)\}$$




ExtractMin returns $b$

# Step 3: DecreaseKey

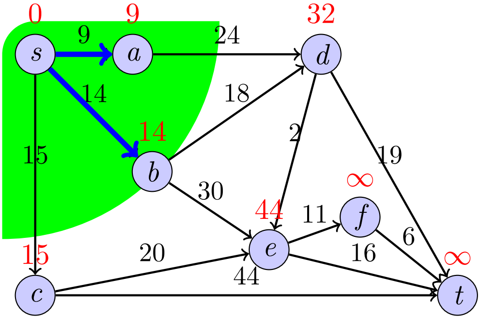
$$S = \{s, a, b\}$$
$$PQ = \{c(15), d(33), e(\infty), f(\infty), t(\infty)\}$$




$\textsc{DecreaseKey}(d, 32)$
$\textsc{DecreaseKey}(e, 44)$

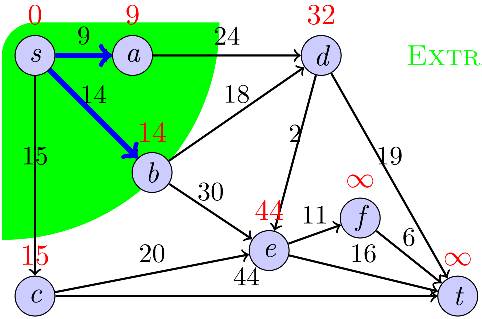
# Step 4: ExtractMin

$$S = \{s, a, b\}$$
$$PQ = \{c(15), d(32), e(44), f(\infty), t(\infty)\}$$



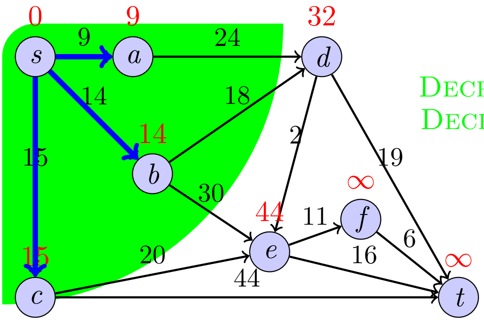
$S = \{s, a, b\}$

$PQ = \{c(15), d(32), e(44), f(\infty), t(\infty)\}$




ExtractMin returns $c$

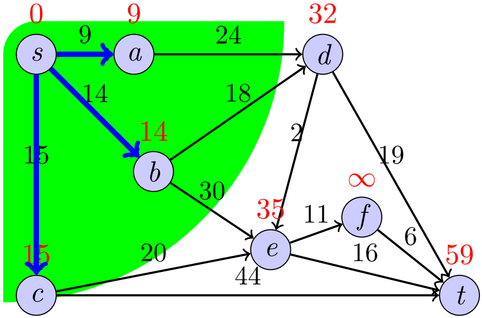
$S = \{s, a, b, c\}$
$PQ = \{d(32), e(44), f(\infty), t(\infty)\}$




DecreaseKey$(e, 35)$
DecreaseKey$(t, 59)$

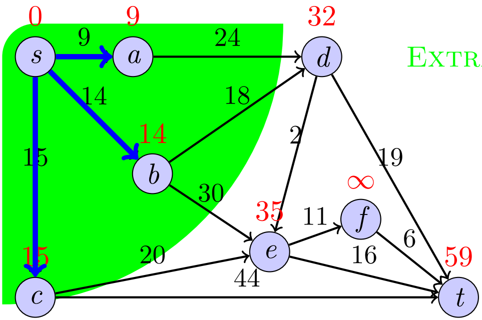
# Step 5: ExtractMin

$$S = \{s, a, b, c\}$$
$$PQ = \{d(32), e(35), t(59), f(\infty)\}$$



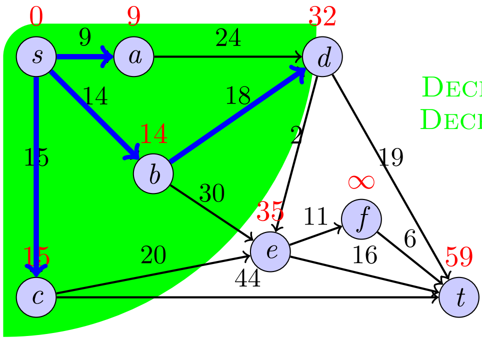
# Step 5: ExtractMin

$S = \{s, a, b, c\}$
$PQ = \{d(32), e(35), t(59), f(\infty)\}$




ExtractMin returns $d$

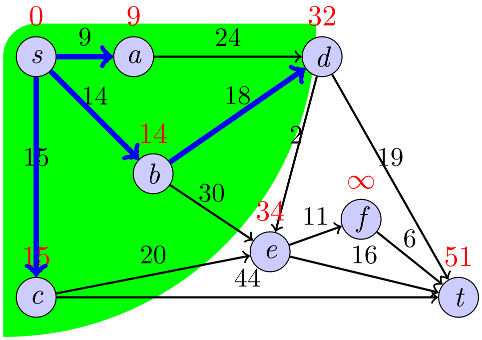
$S = \{s, a, b, c, d\}$
$PQ = \{e(35), t(59), f(\infty)\}$




DECREASEKEY$(t, 51)$
DECREASEKEY$(e, 34)$

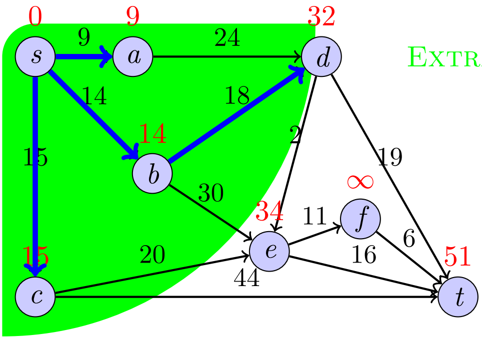
# Step 6: ExtractMin

$S = \{s, a, b, c, d\}$
$PQ = \{e(34), t(51), f(\infty)\}$



# Step 6: ExtractMin

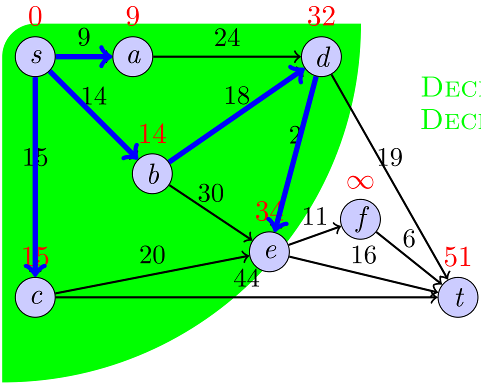
$$S = \{s, a, b, c, d\}$$
$$PQ = \{e(34), t(51), f(\infty)\}$$




ExtractMin returns $e$

# Step 6: DecreaseKey

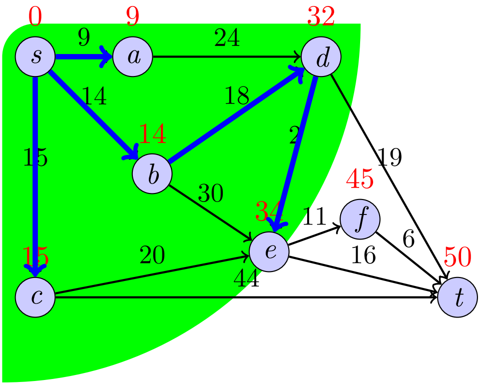
$S = \{s, a, b, c, d, e\}$
$PQ = \{f(45), t(50)\}$




DecreaseKey$(f, 45)$
DecreaseKey$(t, 50)$

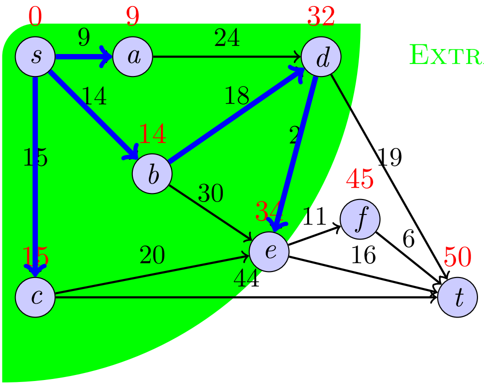
# Step 7: ExtractMin

$S = \{s, a, b, c, d, e\}$
$PQ = \{f(45), t(50)\}$



# Step 7: ExtractMin

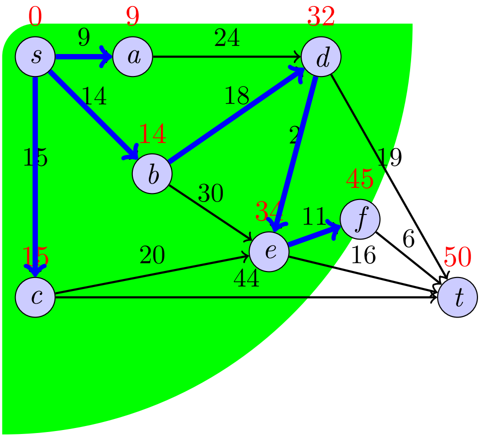
$S = \{s, a, b, c, d, e\}$
$PQ = \{f(45), t(50)\}$



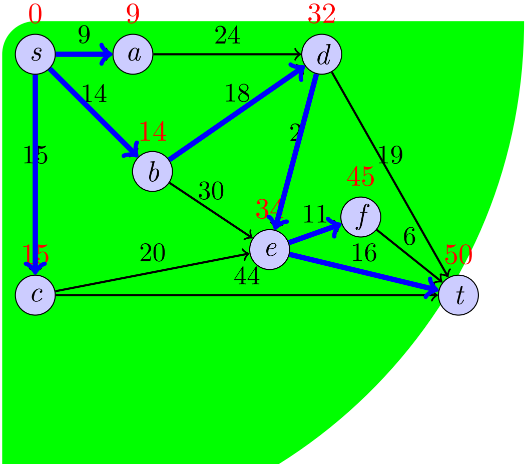

ExtractMin returns $f$

# Step 7: DecreaseKey

$S = \{s, a, b, c, d, e, f\}$
$PQ = \{t(50)\}$



$$S = \{s, a, b, c, d, e, f, t\}$$
$$PQ = \{\}$$



# Time complexity of DIJKSTRA algorithm

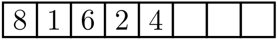
| Operation | Linked list | Binary heap | Binomial heap | Fibonacci heap |
|---|---|---|---|---|
| MAKEHEAP | 1 | 1 | 1 | 1 |
| INSERT | 1 | $\log n$ | $\log n$ | 1 |
| EXTRACTMIN | $n$ | $\log n$ | $\log n$ | $\log n$ |
| DECREASEKEY | 1 | $\log n$ | $\log n$ | 1 |
| DELETE | $n$ | $\log n$ | $\log n$ | $\log n$ |
| UNION | 1 | $n$ | $\log n$ | 1 |
| FINDMIN | $n$ | 1 | $\log n$ | 1 |
| DIJKSTRA | $O(n^2)$ | $O(m \log n)$ | $O(m \log n)$ | $O(m + n \log n)$ |

DIJKSTRA algorithm: $n$ INSERT, $n$ EXTRACTMIN, and $m$ DECREASEKEY.

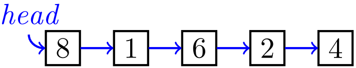
Implementing priority queue: array or linked list

# Implementing priority queue: unsorted array

- Unsorted array:



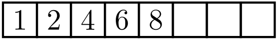

- Unsorted linked list:



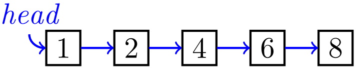

- Operations:
  - INSERT: $O(1)$
  - EXTRACTMIN: $O(n)$
- Note: a list containing $n$ elements is too long to find the minimum efficiently.

# Implementing priority queue: sorted array

- Sorted array:




- Sorted linked list:




- Operations:
  - INSERT: $O(n)$
  - EXTRACTMIN: $O(1)$
- Note: a list containing $n$ elements is too long to maintain the order among elements.

# Implementing priority queue: array or linked list

| Operation | Linked List |
| --- | --- |
| INSERT | $O(1)$ |
| EXTRACTMIN | $O(n)$ |
| DECREASEKEY | $O(1)$ |
| UNION | $O(1)$ |

Binary heap: from a linked list to a tree

# Binary heap

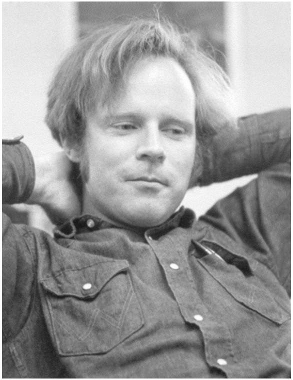
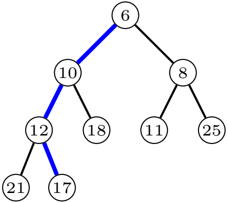


Figure 1: R. W. Floyd [1964]

# Binary heap: a complete binary tree

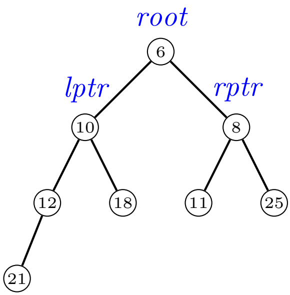
- Basic idea:
  - **loosing the structure**: Recall that the objective is to find the minimum. To achieve this objective, it is not necessary to sort all elements.
  - **but don't loose it too much**: we still need order between some elements.




- Binary heap: elements are stored in a **complete binary tree**, i.e., a tree that is perfectly balanced except for the bottom level. **Heap order** is required, i.e., any parent has a key smaller than his children;
- Advantage: any path has a short length of $O(\log_2 n)$ rather than $n$ in linked list, making it efficient to maintain heap order.

# Binary heap: an explicit implementation

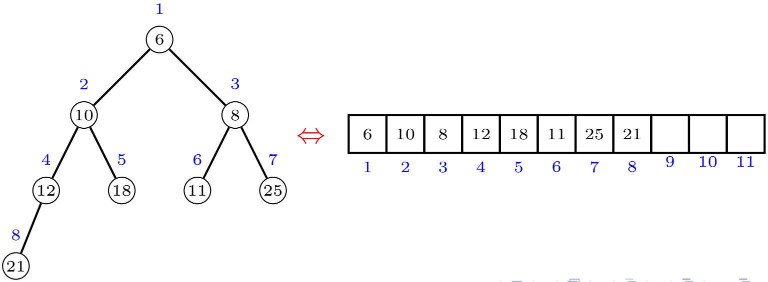
- **Pointer representation**: each node has pointers to its parent and two children;
- The following information are maintained:
  - the number of elements $n$;
  - the pointer to the root node;




- Note: the last node can be found in $O(\log n)$ time.

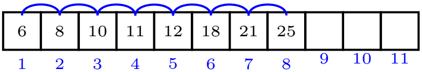
# Binary heap: an implicit implementation

- **Array representation**: one-one correspondence between a binary tree and an array.
  - Binary tree $\Rightarrow$ array:
    - the indices starting from $1$ for the sake of simplicity;
    - the indices record the order that the binary tree is traversed **level by level**.
  - Array $\Rightarrow$ binary tree:
    - the $k$-th item has two children located at $2k$ and $2k+1$;
    - the parent of the $k$-th item is located at $\lfloor \frac{k}{2} \rfloor$;

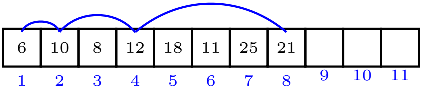


# Sorted array vs. binary heap

- Sorted array: an array containing $n$ elements in an increasing order;




- Binary heap: heap order means that only the order among nodes in short paths (length is less than $\log n$) are maintained. Note that some inverse pairs exist in the array.



## Binary heap: primitive and other operations

# Primitive: exchanging nodes to restore heap order

- Primitive operation: when heap order is violated, i.e. a parent has a value larger than only one of its children, we simply exchange them to resolve the conflict.

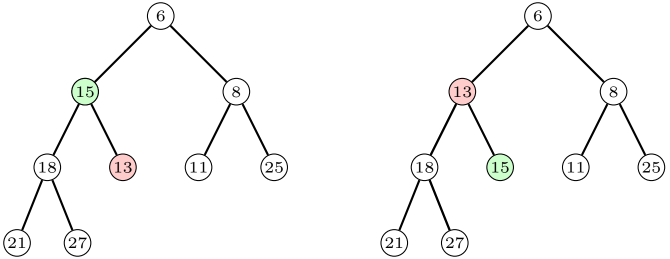



Figure 2: Heap order is violated: $15 > 13$. Exchange them to resolve the conflict.

# Primitive: exchanging nodes to restore heap order

- Primitive operation: when heap order is violated, i.e. a parent has a value larger than both of its children, we exchange the parent with its smaller child to resolve the conflict.

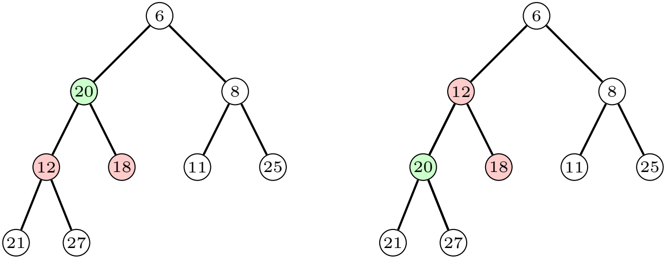
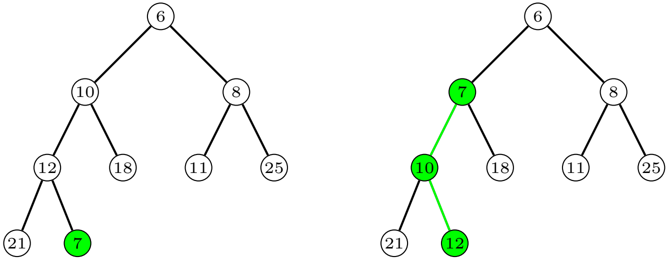


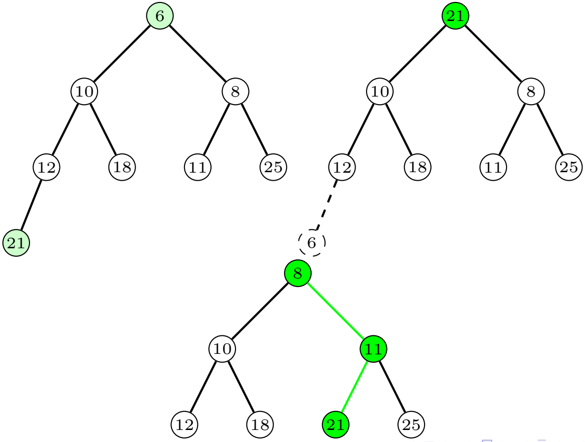

Figure 3: Heap order is violated: $20 > 12$, and $20 > 18$. Exchange 20 with its smaller child (12) to resolve the conflicts.

- INSERT operation: the element is added as a new node at the end. Since the heap order might be violated, the node is repeatedly exchanged with its parent until heap order is restored.
- For example, INSERT( 7 ):



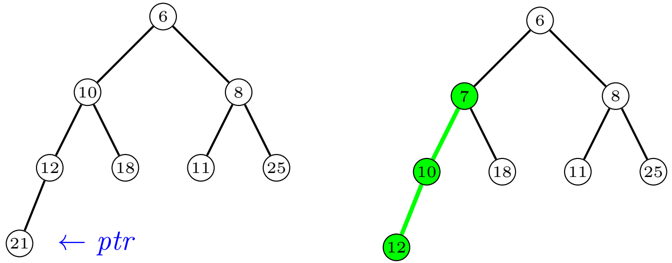
# Binary heap: ExtractMin operation

- ExtractMin operation: exchange element in root with the last node; repeatedly exchange the element in root with its **smaller child** until heap order is restored.
- For example, ExtractMin():



# Binary heap: DECREASEKEY operation

- DECREASEKEY operation: given a handle to a node, repeatedly exchange the node with its parent until heap order is restored.
- For example, DECREASEKEY( $ptr$, 7 ):



# Binary heap: analysis

**Theorem**

*In an* **implicit** *binary heap, any sequence of $m$* INSERT, . DECREASEKEY, *and* EXTRACTMIN *operations with $n$* INSERT *operations takes $O(m \log n)$ time.*

Note:

- Each operation touches at most $\log n$ nodes on a path from the root to a leaf.

**Theorem**

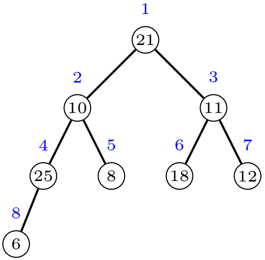
*In an* **explicit** *binary heap with $n$ nodes, the* INSERT, . DECREASEKEY, *and* EXTRACTMIN *operations take $O(m \log n)$ time in the worst case.*

Note:

- If using array representation, a dynamic array expanding/contracting is needed. However, the total cost of array expanding/contracting is $O(n)$ (see TABLEINSERT).

# Binary heap: heapify a set of items

- Question: Given a set of $n$ elements, how to construct a binary heap containing them?
- Solutions:
  1. Simply INSERT the elements one by one. Takes $O(n \log n)$ time.
  2. Bottom-up heapifying. Takes $O(n)$ time.
     For $i = n$ to $1$, we repeatedly exchange the element in node $i$ with its smaller child until the subtree rooted at node $i$ is heap-ordered.




| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 21 | 10 | 11 | 25 | 8 | 18 | 12 | 6 | | | |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |

(see a demo)

# Binary heap: heapify

## Theorem

*Given $n$ elements, a binary heap can be constructed using $O(n)$ time.*

## Proof.

- There are at most $\lceil \frac{n}{2^{h+1}} \rceil$ nodes of height $h$;
- It takes $O(h)$ time to sink a node of height $h$;
- The total time is:

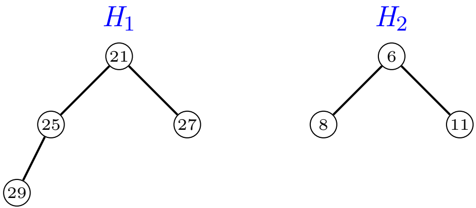
$$\sum_{h=0}^{\lfloor \log_2 n \rfloor} \lceil \frac{n}{2^{h+1}} \rceil h \le \sum_{h=0}^{\lfloor \log_2 n \rfloor} n \frac{h}{2^h} \le 2n$$

□

# Implementing priority queue: binary heap

| Operation | Linked List | Binary Heap |
|---|---|---|
| Insert | $O(1)$ | $O(\log n)$ |
| ExtractMin | $O(n)$ | $O(\log n)$ |
| DecreaseKey | $O(1)$ | $O(\log n)$ |
| Union | $O(1)$ | $O(n)$ |

- UNION operation: Given two binary heaps $H_1$ and $H_2$, to merge them into one binary heap.




- $O(n)$ time is needed if using heapify.
- Question: Is there a quicker way to union two heaps?

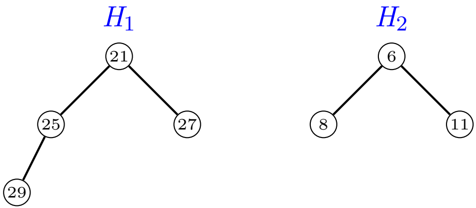
Binomial heap: using multiple trees rather than a single tree to support efficient UNION operation

# Binomial heap

Figure 4: Jean Vuillenmin [1978]

# Binomial heap: efficient UNION

- Basic idea:
  - **loosing the structure**: if **multiple trees** are allowed to represent a heap, UNION can be efficiently implemented via simply putting trees together.
  - **but don't loose it too much**: there should not be too many trees; otherwise, it will take a long time to find the minimum among all root nodes.

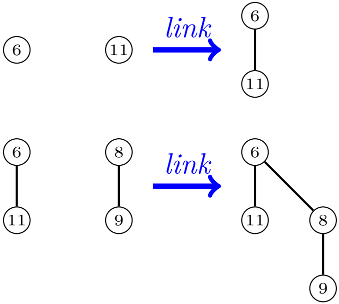



- EXTRACTMIN: simply finding the minimum element of the root nodes. Note that a root node holds the minimum of the tree due to the heap order.

# Why we can't loose the structure too much?

- An extreme case of multiple trees: each node is itself a tree. Then it will take $O(n)$ time to find the minimum.

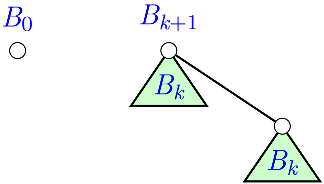

- Solution: **consolidating**, i.e., two trees (with the same size) are merged into one — the larger root is linked to the smaller one to keep the heap order. Note that after consolidating, at most $\log n$ trees will be left.



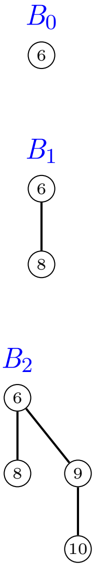
# Binomial tree

**Definition (Binomial tree)**

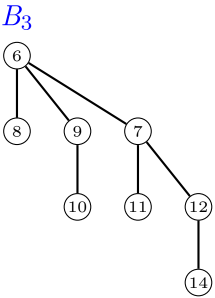
The binomial tree is defined recursively: a single node is itself a $B_0$ tree, and two $B_k$ trees are linked into a $B_{k+1}$ tree.



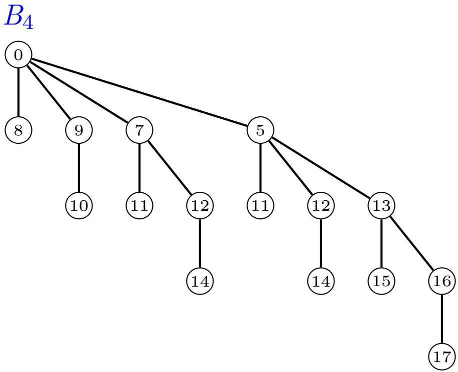
# Binomial tree examples: $B_0$, $B_1$, $B_2$



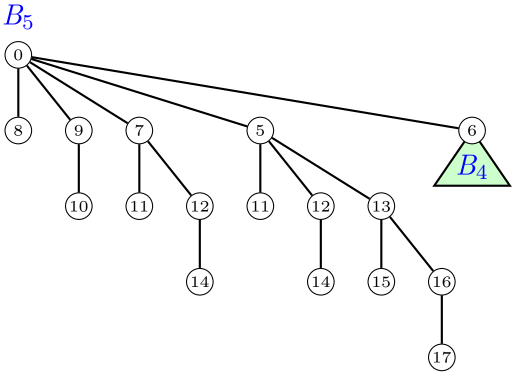
# Binomial tree example: $B_3$



# Binomial tree example: $B_4$

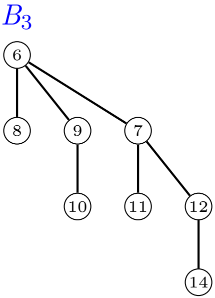


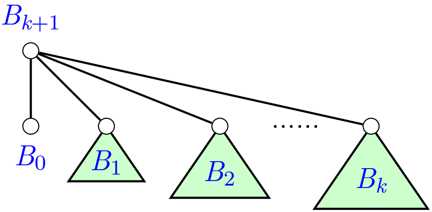
# Binomial tree example: $B_5$



Properties:

1. $|B_k| = 2^k$.
2. $height(B_k) = k$.
3. $degree(B_k) = k$.
4. The $i$-th child of a node has a degree of $i - 1$.
5. The deletion of the root yields trees $B_0$, $B_1$, ..., $B_{k-1}$.
6. Binomial tree is named after the fact that the node number of all levels are binomial coefficients.



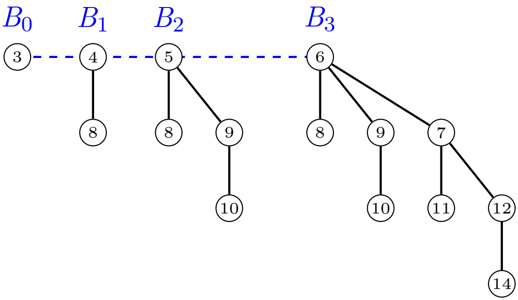
# Binomial heap: a forest

## Definition (Binomial forest)

A binomial heap is a collection of several binomial trees:

- Each tree is heap ordered;
- There is either 0 or 1 $B_k$ for any $k$.

- Example:



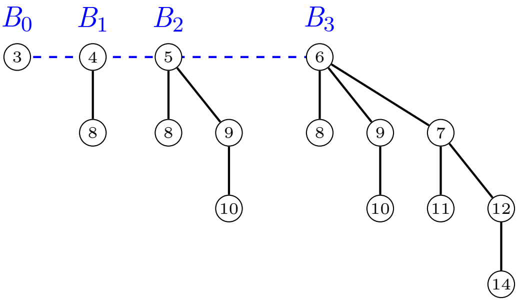

- Note that the roots are organized using doubly-linked circular list, and the minimum of them is recorded using a pointer.

# Binomial heap: properties

Properties:

1. A binomial heap with $n$ nodes contains the binomial tree $B_i$ iff $b_i = 1$, where $b_k b_{k-1} ... b_1 b_0$ is binary representation of $n$.
2. It has at most $\lfloor \log_2 n \rfloor + 1$ trees.
3. Its height is at most $\lfloor \log_2 n \rfloor$.

Thus, it takes $O(\log n)$ time to find the minimum element via checking the roots.



# UNION is efficient: example 1

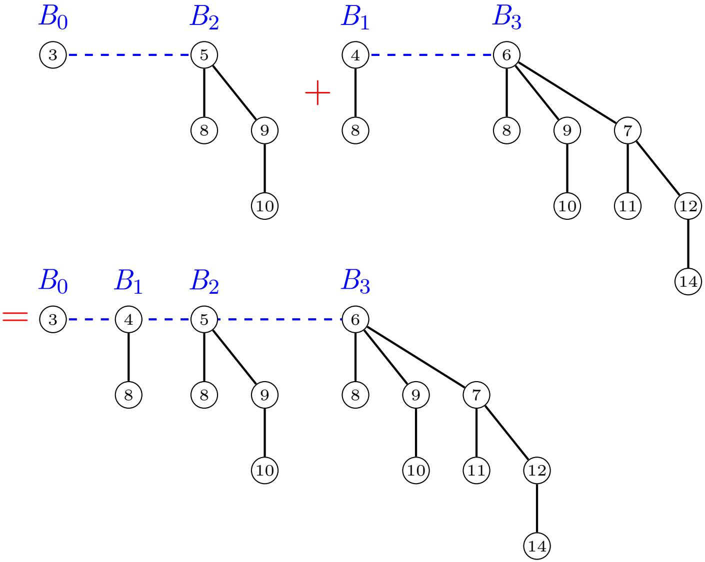



Figure 5: An easy case: no consolidating is needed

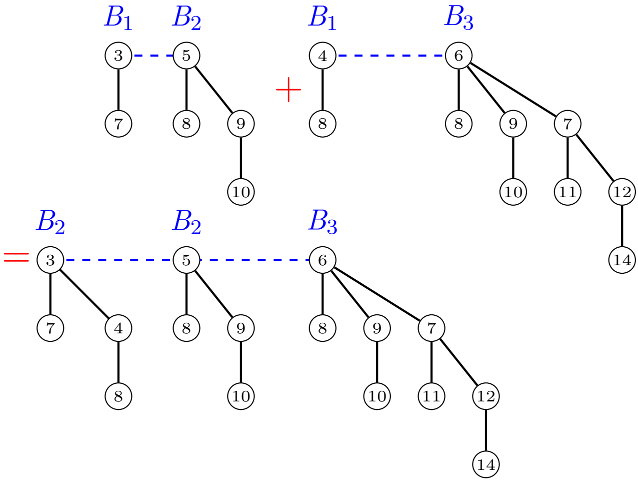



Figure 6: Consolidating two $B_1$ trees into a $B_2$ tree

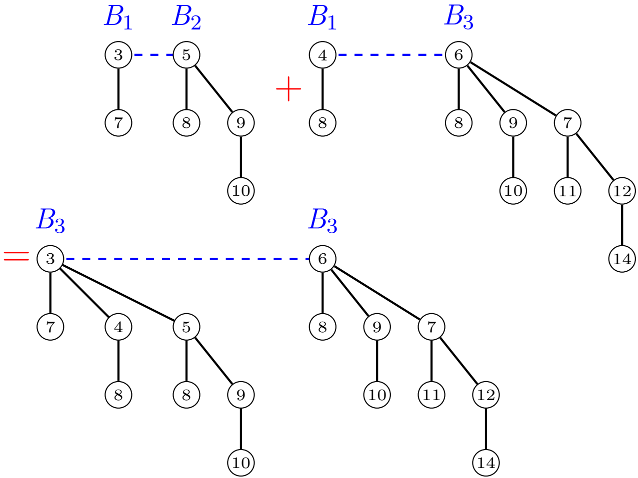
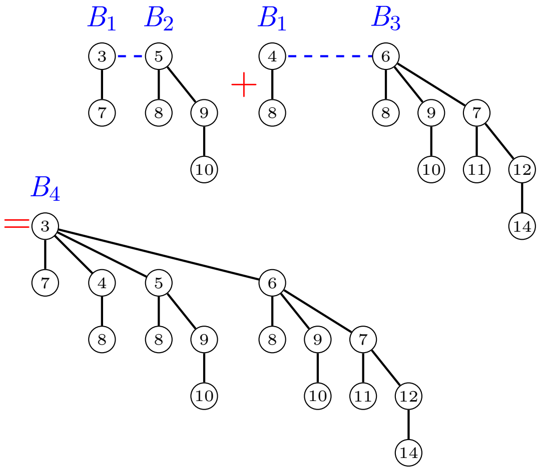



Figure 7: Consolidating two $B_2$ trees into a $B_3$ tree

# UNION is efficient: example 2 III




Time complexity: $O(\log n)$ since there are at most $O(\log n)$ trees.

# Binomial heap: INSERT operation

INSERT($x$)

```
Create a B_0 tree for x;
Change the pointer to the minimum root node if necessary;
while there are two B_k trees for some k do
    Link them together into one B_{k+1} tree;
    Change the pointer to the minimum root node if necessary;
end while
```

# INSERT operation: an example

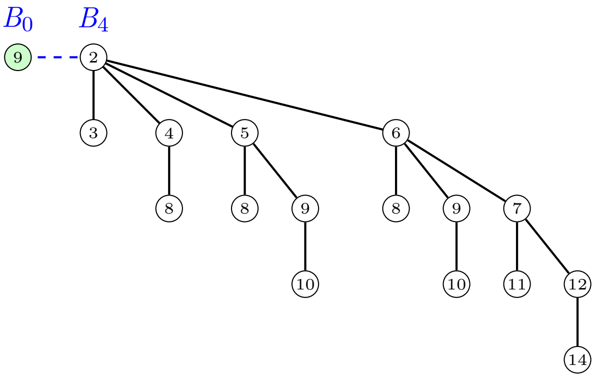



Figure 8: An easy case: no consolidating is needed

# INSERT operation: example 2 I

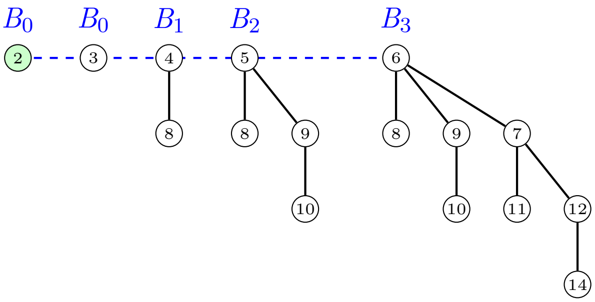



Figure 9: Consolidating two $B_0$

# INSERT operation: example 2 II

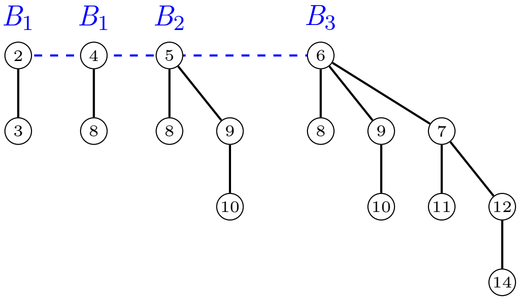



Figure 10: Consolidating two $B_1$

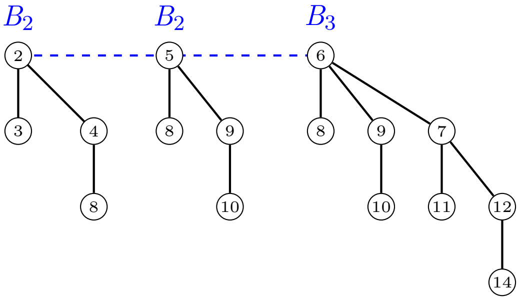



Figure 11: Consolidating two $B_2$

# INSERT operation: example 2 IV

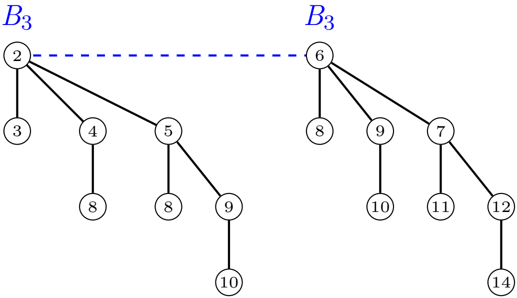
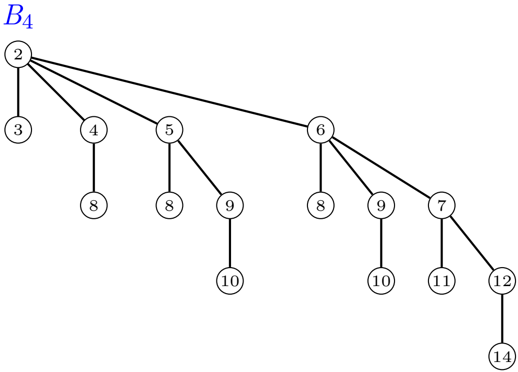



Figure 12: Consolidating two $B_3$

# INSERT operation: example 2 V



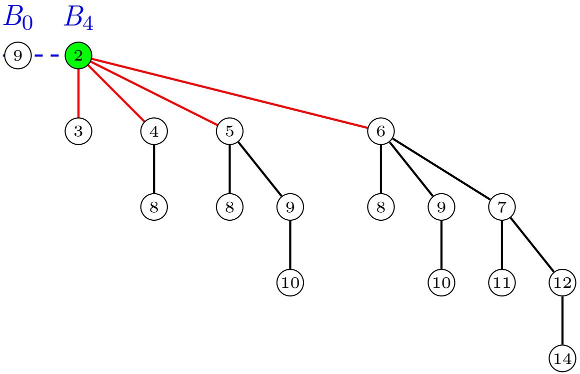

Time complexity: $O(\log n)$ (worst case) since there are at most $\log n$ trees.

# Binomial heap: ExtractMin operation

ExtractMin()

```
Remove the min node, and insert its children into the root list;
Change the pointer to the minimum root node if necessary;
while there are two B_k trees for some k do
    Link them together into one B_{k+1} tree;
    Change the pointer to the minimum root node if necessary;
end while
```

# ExtractMin operation: an example I



# ExtractMin operation: an example II

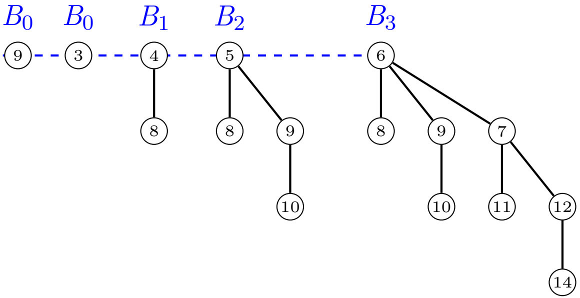



Figure 13: The four children become trees

# ExtractMin operation: an example III

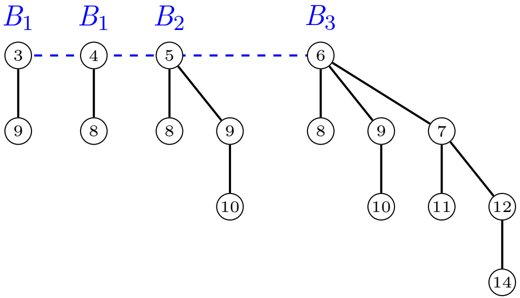



Figure 14: Consolidating two $B_1$ trees

# ExtractMin operation: an example IV

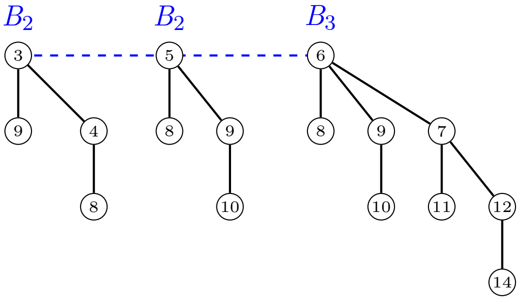



Figure 15: Consolidating two $B_2$ trees

# ExtractMin operation: an example V

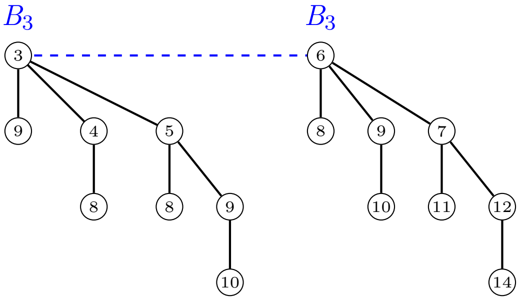
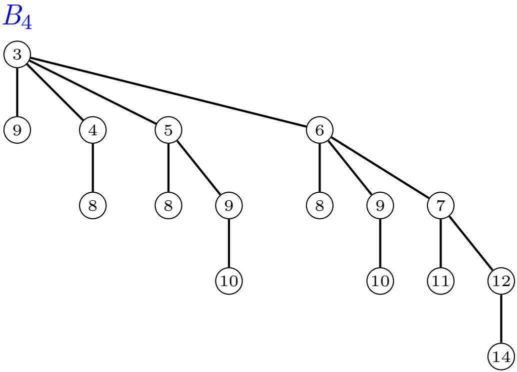



Figure 16: Consolidating two $B_2$ trees




Time complexity: $O(\log n)$

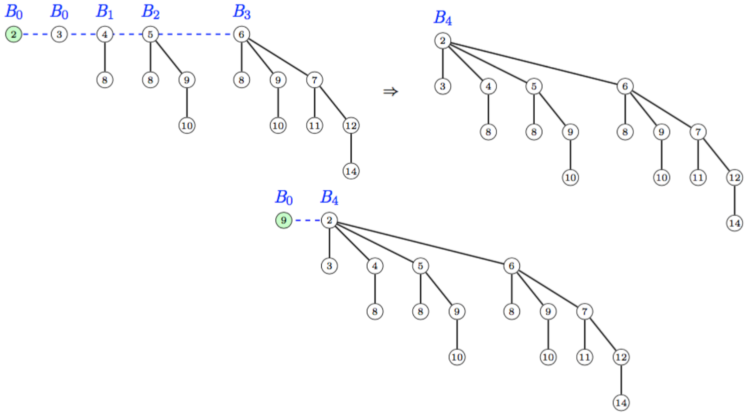
# Implementing priority queue: Binomial heap

| Operation | Linked List | Binary Heap | Binomial Heap |
|---|---|---|---|
| INSERT | $O(1)$ | $O(\log n)$ | $O(\log n)$ |
| EXTRACTMIN | $O(n)$ | $O(\log n)$ | $O(\log n)$ |
| DECREASEKEY | $O(1)$ | $O(\log n)$ | $O(\log n)$ |
| UNION | $O(1)$ | $O(n)$ | $O(\log n)$ |

Binomial heap: more accurate analysis using the amortized technique

Motivation:

- If an INSERT takes a long time (say $\log n$), the subsequent INSERT operations shouldn't take long!




- Thus, it will be more accurate to examine a sequence of operations rather than each operation individually.

# Amortized analysis of INSERT operation

INSERT($x$)

```
Create a B_0 tree for x;
Change the pointer to the minimum root node if necessary;
while there are two B_k trees for some k do
    Link them together into one B_{k+1} tree;
    Change the pointer to the minimum root node if necessary;
end while
```

Analysis:

- A single INSERT operation takes time $1 + w$, where $w = \#$WHILE.
- For the sake of calculating the total running time of a sequence of operations, we represent the running time of a single operation as decrease of a potential function.
- Consider a quantity $\Phi = \#trees$ (called potential function). The changes of $\Phi$ during an operation are:
  - $\Phi$ increase: $1$.
  - $\Phi$ decrease: $w$.
- Thus the running time of INSERT can be rewritten in terms of $\Phi$ as $1 + w = 1 +$ decrease in $\Phi$. Note that this representation makes it convenient to sum running time of a sequence of INSERT operations.

# Amortized analysis of ExtractMin

ExtractMin()

```
Remove the min node, and insert its children to the root list;
Change the pointer to the minimum root node if necessary;
while there are two B_k trees for some k do
    Link them together into one B_{k+1} tree;
    Change the pointer to the minimum root node if necessary;
end while
```

Analysis:

- A single ExtractMin operation takes $d + w$ time, where $d$ denotes degree of the removed root node, and $w = \#\texttt{WHILE}$.
- For the sake of calculating the total running time of a sequence of operations, we represent the running time of a single operation as decrease of a potential function.
- Consider a potential function $\Phi = \#trees$. The changes during an operation are:
  - $\Phi$ increase: $d$.
  - $\Phi$ decrease: $w$.
- Similarly, the running time is rewritten in terms of $\Phi$ as $d + w = d+$ decrease in $\#trees$. Note that $d \leq \log n$.

# Amortized analysis

- Let's consider any sequence of $n$ INSERT and $m$ EXTRACTMIN operations.
- The total running time is at most $n + m \log n +$ total decrease in $\#trees$.
- Note: total decrease in $\#trees \leq$ total increase in $\#trees$ (why?), which is at most $n + m \log n$.
- Thus the total time is at most $2n + 2m \log n$.
- We say INSERT takes $O(1)$ amortized time, and EXTRACTMIN takes $O(\log n)$ amortized time.

**Definition (Amortized time)**

For any sequence of $n_1$ operation 1, $n_2$ operation 2..., if the total time is $O(n_1 T_1 + n_2 T_2 ...)$, we say that operation 1 takes $T_1$ amortized time, operation 2 takes $T_2$ amortized time ....

# Intuition of the amortized analysis

- The actual running time of an $\textsc{Insert}$ operation is $1 + w$. A large $w$ means that the $\textsc{Insert}$ operation takes a long time. Note that the $w$ time was spent on "decreasing trees"; thus, if the $w$ time was amortized over the operations "creating trees", the "amortized time" of $\textsc{Insert}$ operation will be only $O(1)$.
- The actual running time of an $\textsc{ExtractMin}$ operation is at most $\log n + w$. Note that at most $\log n$ new trees are created during an $\textsc{ExtractMin}$ operation; thus, the amortized time is still $O(\log n)$ even if some costs have been amortized to it from other operations due to "tree creating".

# Implementing priority queue: Binomial heap

| Operation | Linked List | Binary Heap | Binomial Heap | Binomial Heap* |
|---|---|---|---|---|
| INSERT | $O(1)$ | $O(\log n)$ | $O(\log n)$ | $O(1)$ |
| EXTRACTMIN | $O(n)$ | $O(\log n)$ | $O(\log n)$ | $O(\log n)$ |
| DECREASEKEY | $O(1)$ | $O(\log n)$ | $O(\log n)$ | $O(\log n)$ |
| UNION | $O(1)$ | $O(n)$ | $O(\log n)$ | $O(1)$ |

*amortized cost

# Binomial heap: DecreaseKey operation

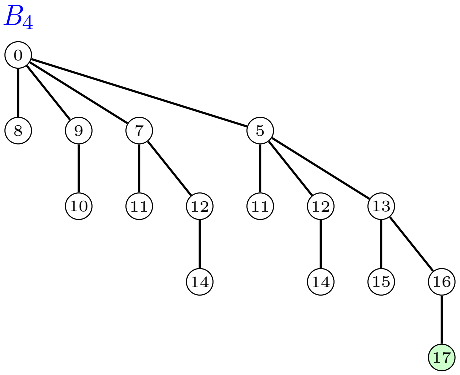



Figure 17: DecreaseKey: 17 to 1

- Time: $O(\log n)$ since in the worst case, we need to perform node exchanging up to the root.
- Question: is there a quicker way for decrease key?

Fibonacci heap: an efficient implementation of DECREASEKEY via simply cutting an edge rather than exchanging nodes

# Fibonacci heap

Figure 18: Robert Tarjan [1986]

# Fibonacci heap: an efficient DECREASEKEY operation

- Basic idea:
  - **loosing the structure**: Binomial heap requires trees to be in perfect shape. Now we loose this restriction — when heap order is violated, a simple solution is to "cut off a node, and insert it into the root list".
  - **but don't loose it too much**: the "cutting off" operation makes a tree not "binomial" any more; however, it should not deviate from a binomial tree too much. A technique to achieve this objective is allowing any non-root node to lose "at most one child".

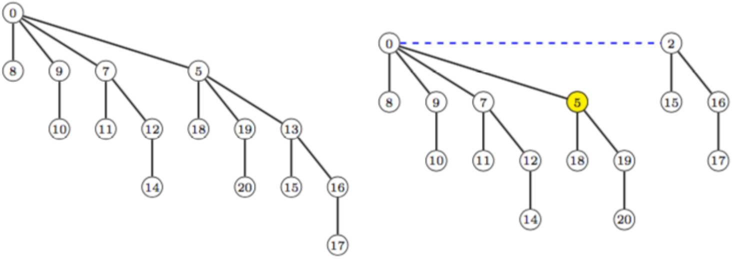



Figure 19: Heap order is violated when DESCREASEKEY 13 to 2.

# Fibonacci heap: DecreaseKey

DecreaseKey$(v, x)$

```
key(v) = x;
if heap order is violated then
    u = v's parent;
    Cut subtree rooted at node v, and insert it into the root list;
    Change the pointer to the minimum root node if necessary;
    while u is marked do
        Cut subtree rooted at node u, and insert it into the root
        list;
        Change the pointer to the minimum root node if
        necessary;
        Unmark u;
        u = u's parent;
    end while
    Mark u;
end if
```

# DecreaseKey: an example I

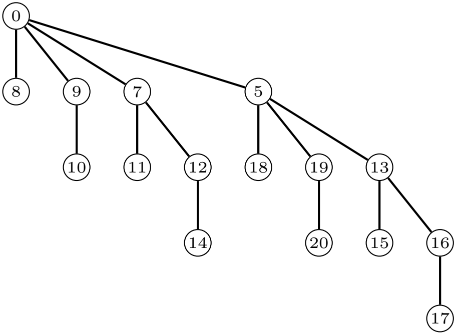



Figure 20: A Fibonacci heap. To DecreaseKey: 19 to 3.

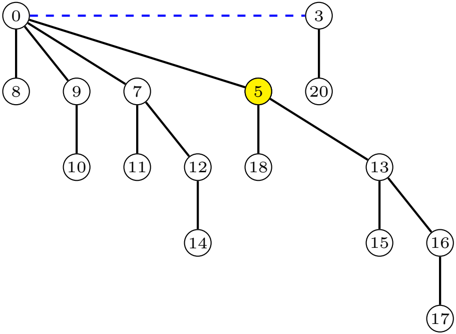



Figure 21: After DecreaseKey: 19 to 3. To DecreaseKey: 15 to 2.

# DecreaseKey: an example III

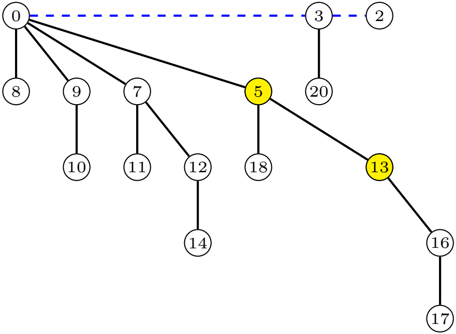



Figure 22: After DecreaseKey: 15 to 2. To DecreaseKey: 12 to 8.

# DecreaseKey: an example IV

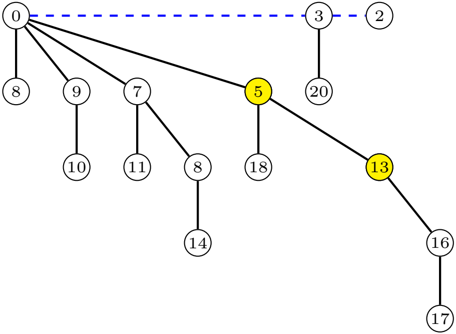



Figure 23: After DecreaseKey: 12 to 8. To DecreaseKey: 14 to 1.

# DecreaseKey: an example V

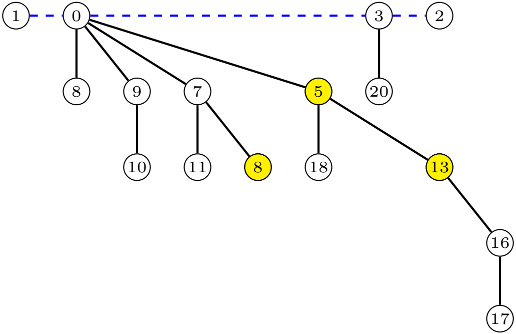



Figure 24: After DecreaseKey: 14 to 1. To DecreaseKey: 16 to 9.

# DecreaseKey: an example VI

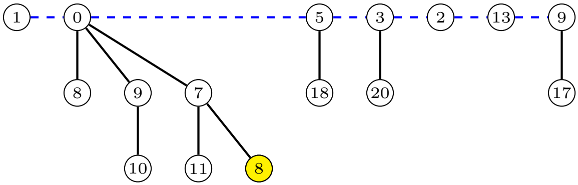



Figure 25: After DecreaseKey: 16 to 9

# Fibonacci heap: INSERT

INSERT($x$)

1: Create a tree for $x$, and insert it into the root list;
2: Change the pointer to the minimum root node if necessary;

Note: **Being lazy!** Consolidating trees when extracting minimum.

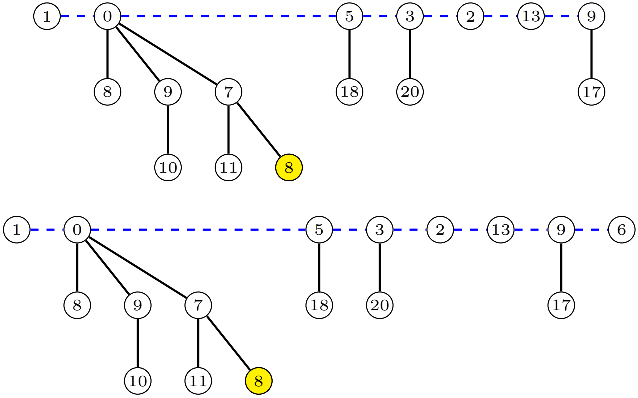



Figure 26: INSERT(6): creating a new tree, and insert it into the root list

# Fibonacci heap: ExtractMin

ExtractMin()

```
Remove the min node, and insert its children into the root list;
Change the pointer to the minimum root node if necessary;
while there are two roots u and v of the same degree do
    Consolidate the two trees together;
    Change the pointer to the minimum root node if necessary;
end while
```

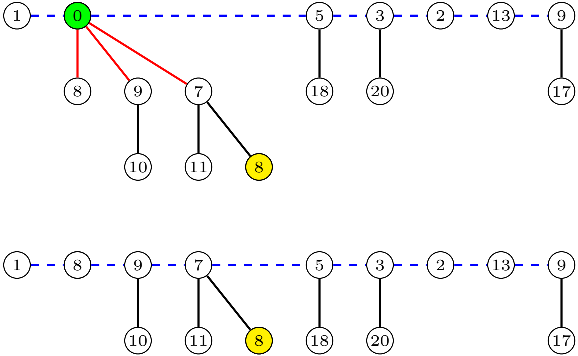



Figure 27: ExtractMin: removing the min node, and adding 3 trees

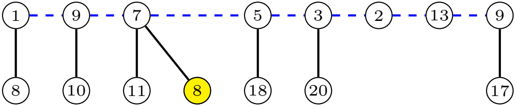



Figure 28: EXTRACTMIN: after consolidating two trees rooted at node 1 and 8

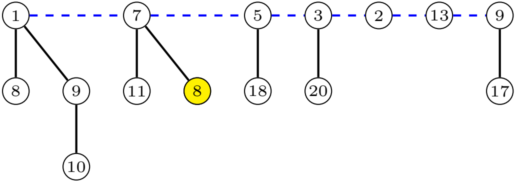



Figure 29: EXTRACTMIN: after consolidating two trees rooted at node 1 and 9

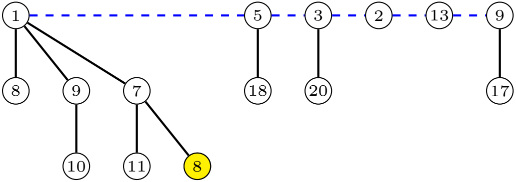



Figure 30: ExtractMin: after consolidating two trees rooted at node 1 and 7

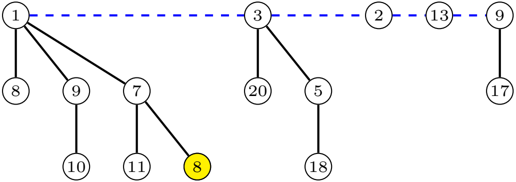



Figure 31: ExtractMin: after consolidating two trees rooted at node 3 and 5

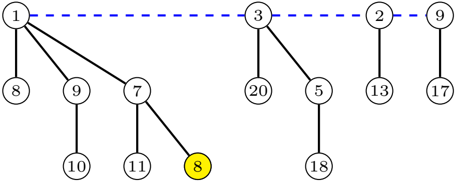



Figure 32: EXTRACTMIN: after consolidating two trees rooted at node 2 and 13

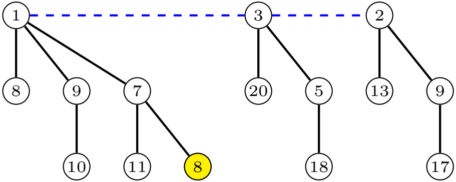



Figure 33: EXTRACTMIN: after consolidating two trees rooted at node 2 and 9

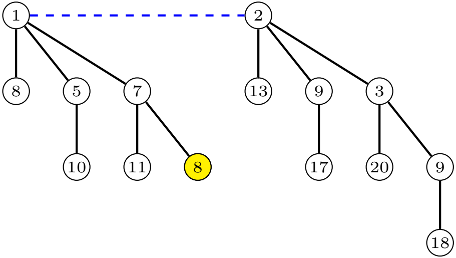



Figure 34: ExtractMin: after consolidating two trees rooted at node 2 and 3

# ExtractMin: an example VI

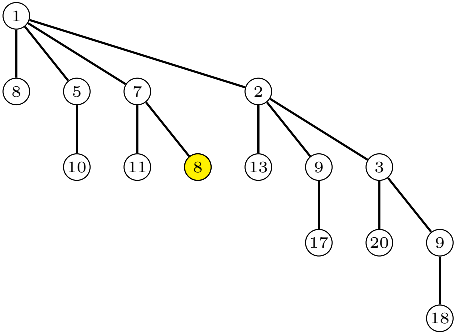



Figure 35: ExtractMin: after consolidating two trees rooted at node 1 and 2

## Fibonacci heap: an amortized analysis

# Fibonacci heap: DecreaseKey

DecreaseKey$(v, x)$

$key(v) = x$;
**if** heap order is violated **then**
  $u = v's$ parent;
  Cut subtree rooted at node $v$, and insert it into the root list;
  Change the pointer to the minimum root node if necessary;
  **while** $u$ is marked **do**
    Cut subtree rooted at node $u$, and insert it into the root list;
    Change the pointer to the minimum root node if necessary;
    Unmark $u$;
    $u = u's$ parent;
  **end while**
  Mark $u$;
**end if**

Analysis:

- The actual running time of a single operation is $1 + w$, where $w = \#\texttt{WHILE}$.
- To calculate the **total running time** of **a sequence of operations**, we represent the running time of **a single operation** as **decrease of a potential function**.
- Consider a **potential function** $\Phi = \#trees + 2\#marks$. The changes of $\Phi$ during an operation are:
  - $\Phi$ increase: $1 + 2 = 3$.
  - $\Phi$ decrease: $(-1 + 2 * 1) * w = w$.
- Thus we can rewrite the running time in terms of $\Phi$ as $1 + w = 1 + \Phi$ decrease.

Intuition: a large $w$ means that DecreaseKey takes a long time; however, if we can "amortize" $w$ over other operations, a DecreaseKey operation takes only $O(1)$ "amortized time".

# Fibonacci heap: ExtractMin

ExtractMin()

```
Remove the min node, and insert its children into the root list;
Change the pointer to the minimum root node if necessary;
while there are two roots u and v of the same degree do
    Consolidate the two trees together;
    Change the pointer to the minimum root node if necessary;
end while
```

# ExtractMin: analysis

Analysis:

- The actual running time of a single operation is $d + w$, where $d$ denotes degree of the removed node, and $w = \#\texttt{WHILE}$.
- To calculate **the total running time** of **a sequence of operations**, we represent the running time of **a single operation** as **decrease of a potential function**.
- Consider a **potential function** $\Phi = \#trees + 2\#marks$. The changes of $\Phi$ during an operation are:
  - $\Phi$ increase: $d$.
  - $\Phi$ decrease: $w$.
- Thus the running time can be rewritten in terms of $\Phi$ as $d + w = d+$ decrease in $\Phi$.

Note: $d \leq d_{max}$, where $d_{max}$ denotes the maximum root node degree.

# Fibonacci heap: Insert

Insert($x$)

Create a tree for $x$, and insert it into the root list;
Change the pointer to the minimum root node if necessary;

Analysis:

- The actual running time is $1$, and the changes of $\Phi$ during this operation are:
  - $\Phi$ increase: 1.
  - $\Phi$ decrease: 0.

Note:

- Recall that a binomial heap consolidates trees in both Insert and ExtractMin operations.
- In contrast, the Fibonacci heap adopts the strategy of **"being lazy"** — tree consolidating is removed from Insert operation for the sake of efficiency, and there is no tree consolidating until an ExtractMin operation.

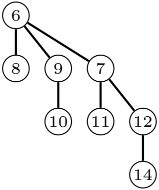
# Fibonacci heap: amortized analysis

- Consider any sequence of $n$ INSERT, $m$ EXTRACTMIN, and $r$ DECREASEKEY operations.
- The total running time is at most: $n + md_{max} + r +$ total decrease in $\Phi$.
- Note: total decrease in $\Phi \leq$ total increase in $\Phi =$ $n + md_{max} + 3r$.
- Thus the total running time is at most: $n + md_{max} + r + n + md_{max} + 3r = 2n + 2md_{max} + 4r$.
- Thus INSERT takes $O(1)$ amortized time, DECREASEKEY takes $O(1)$ amortized time, and EXTRACTMIN takes $O(d_{max})$ amortized time.
- In fact, EXTRACTMIN takes $O(\log n)$ amortized time since $d_{max}$ can be upper-bounded by $\log n$ (why?).

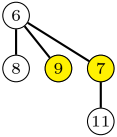
# Fibonacci heap: bounding $d_{max}$

# Fibonacci heap: bounding $d_{max}$

- Recall that for a binomial tree having $n$ nodes, the root degree $d$ is **exactly** $\log_2 n$, i.e. $d = \log_2 n$.



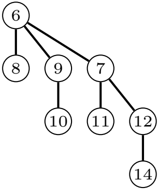

- In contrast, a tree in a Fibonacci heap might have several subtrees cutting off, leading to $d \geq \log_2 n$.



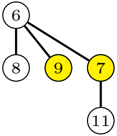

- However, the "marking technique" guarantees that any node can lose at most one child, thus limiting the deviation from the original binomial tree, i.e. $\log_\phi n \geq d \geq \log_2 n$, where $\phi = \frac{1+\sqrt{5}}{2} = 1.618$

# Fibonacci heap: a property of node degree

- Recall that for a binomial tree, the $i$-th child of each node has a degree of exactly $i - 1$.

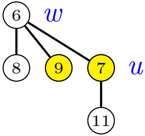



- For a tree in a Fibonacci heap , we will show that the $i$-th child of each node has degree $\geq i - 2$.



## Lemma

*For any node in a Fibonacci heap, the $i$-th child has a degree $\geq i-2$.*




## Proof.

- Suppose $u$ is the **current** $i$-th child of $w$;
- If $w$ is not a root node, it has at most 1 child lost; otherwise, it might have multiple children lost;
- Consider the time when $u$ is linked to $w$. At that time, $degree(w) \geq i-1$, so $degree(u) = degree(w) \geq i-1$;
- Subsequently, $degree(u)$ decreases by at most 1 (Otherwise, $u$ will be cut off and no longer a child of $w$).
- Thus, $degree(u) \geq i-2$.

- Let $F_k$ be **the smallest tree** with root degree of $k$, and for any node of $F_k$, the $i$-th child has degree $\geq i - 2$;


Figure 36: $|B_1| = 2^1$ and $|F_0| = 1 \geq \phi^0$

# Example: $B_2$ versus $F_1$

- Let $F_k$ be the smallest tree with root degree of $k$, and for any node of $F_k$, the $i$-th child has degree $\geq i - 2$;


Figure 37: $|B_2| = 2^2$ and $|F_1| = 2 \geq \phi^1$

# Example: $B_3$ versus $F_2$

- Let $F_k$ be the smallest tree with root degree of $k$, and for any node of $F_k$, the $i$-th child has degree $\geq i-2$;

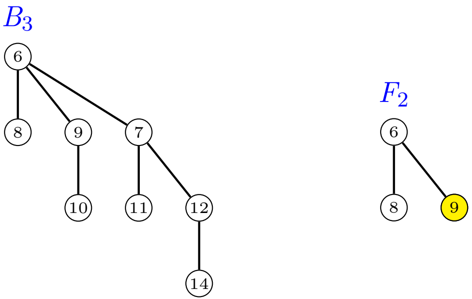



Figure 38: $|B_3| = 2^3$ and $|F_2| = 3 \geq \phi^2$

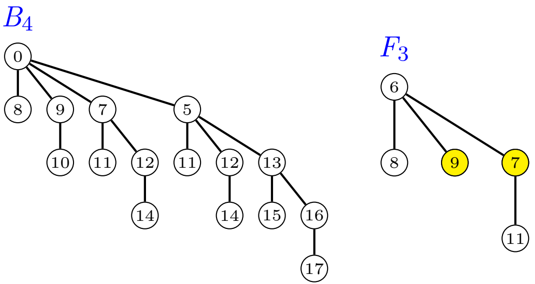



Figure 39: $|B_4| = 2^4$ and $|F_3| = 5 \geq \phi^3$

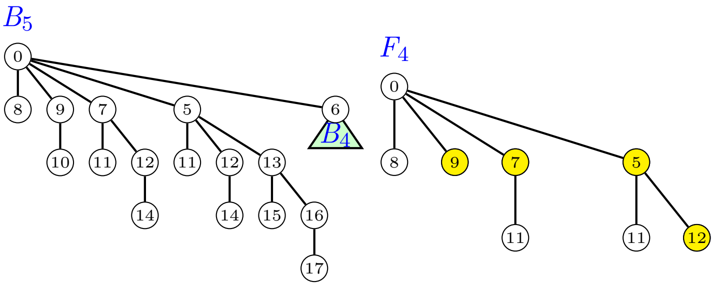
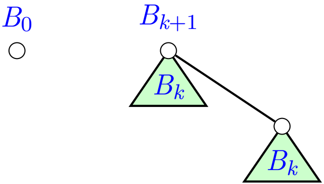


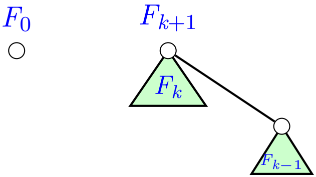

Figure 40: $|B_5| = 2^5$ and $|F_4| = 8 \geq \phi^4$

# General case of trees in Fibonacci heap

- Recall that a binomial tree $B_{k+1}$ is a combination of two $B_k$ trees.




- In contrast, $F_{k+1}$ is the combination of an $F_k$ tree and an $F_{k-1}$ tree.




- We will show that though $F_k$ is smaller than $B_k$, the difference is not too much. In fact, $|F_k| \geq 1.618^k$.

# Fibonacci numbers and Fibonacci heap

## Definition (Fibonacci numbers)

The Fibonacci sequence is $0, 1, 1, 2, 3, 5, 8, 13, 21, 34 \ldots$. It can be defined by the recursion relation: $f_k = \begin{cases} 0 & \text{if } k = 0 \\ 1 & \text{if } k = 1 \\ f_{k-1} + f_{k-2} & \text{if } k \geq 2 \end{cases}$

- Recall that $f_{k+2} \geq \phi^k$, where $\phi = \frac{1+\sqrt{5}}{2} = 1.618\ldots$.
- Note that $|F_k| = f_{k+2}$, say $|F_0| = f_2 = 1$, $|F_1| = f_3 = 2$, $|F_2| = f_4 = 3$.
- Consider a Fibonacci heap $H$ having $n$ nodes. Let $T$ denote a tree in $H$ with root degree $d$.
- We have $n \geq |T| \geq |F_d| = f_{d+2} \geq \phi^d$.
- Thus $d = O(\log_\phi n) = O(\log n)$. So, $d_{max} = O(\log n)$.

Therefore, ExtractMin operation takes $O(\log n)$ amortized time.

# Implementing priority queue: Fibonacci heap

| Operation | Linked List | Binary Heap | Binomial Heap | Binomial Heap* | Fibonacci Heap* |
|---|---|---|---|---|---|
| INSERT | $O(1)$ | $O(\log n)$ | $O(\log n)$ | $O(1)$ | $O(1)$ |
| EXTRACTMIN | $O(n)$ | $O(\log n)$ | $O(\log n)$ | $O(\log n)$ | $O(\log n)$ |
| DECREASEKEY | $O(1)$ | $O(\log n)$ | $O(\log n)$ | $O(\log n)$ | $O(1)$ |
| UNION | $O(1)$ | $O(n)$ | $O(\log n)$ | $O(1)$ | $O(1)$ |

*amortized cost

# Time complexity of DIJKSTRA algorithm

| Operation | Linked list | Binary heap | Binomial heap | Fibonacci heap |
|---|---|---|---|---|
| MAKEHEAP | $1$ | $1$ | $1$ | $1$ |
| INSERT | $1$ | $\log n$ | $\log n$ | $1$ |
| EXTRACTMIN | $n$ | $\log n$ | $\log n$ | $\log n$ |
| DECREASEKEY | $1$ | $\log n$ | $\log n$ | $1$ |
| DELETE | $n$ | $\log n$ | $\log n$ | $\log n$ |
| UNION | $1$ | $n$ | $\log n$ | $1$ |
| FINDMIN | $n$ | $1$ | $\log n$ | $1$ |
| DIJKSTRA | $O(n^2)$ | $O(m \log n)$ | $O(m \log n)$ | $O(m + n \log n)$ |

DIJKSTRA algorithm: $n$ INSERT, $n$ EXTRACTMIN, and $m$ DECREASEKEY.