# Public Key Cryptography

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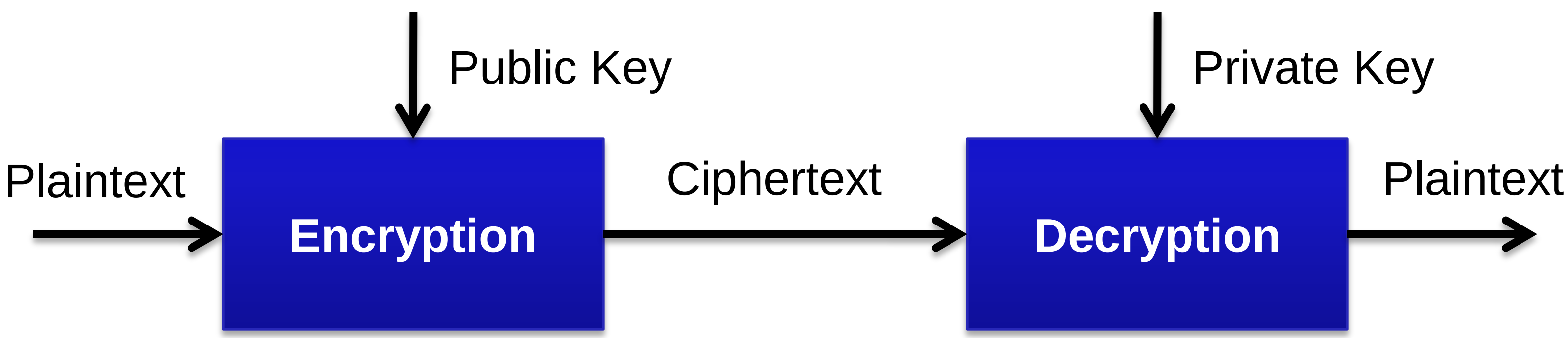
- Symmetric Key:
  - Same key used for encryption and decrypiton
  - Same key used for message integrity and validation
- Public-Key Cryptography
  - Use one key to encrypt or sign messages
  - Use another key to decrypt or validate messages
- Keys
  - Public key known to the world and used to send you a message
  - Only your private key can decrypt the message

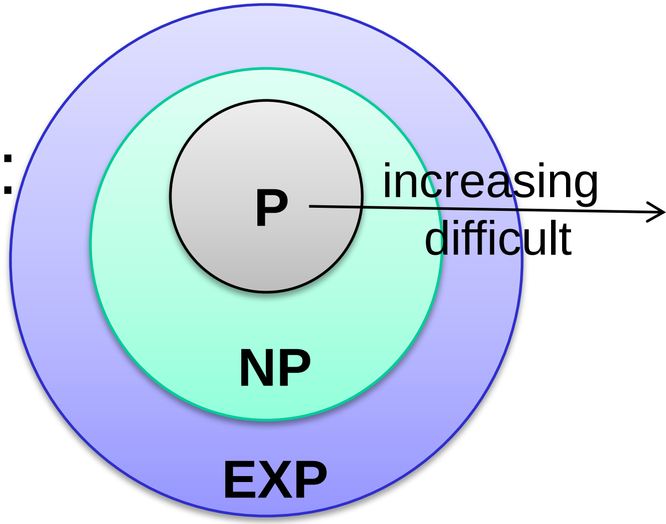


# Public Key Cryptography

- Motivations
  - In symmetric key cryptography, a key was needed between every pair of users wishing to securely communicate
    - $O(n^2)$ keys
  - Problem of establishing a key with remote person with whom you wish to communicate
- Advantages to Public Key Cryptography
  - Key distribution much easier: everyone can known your public key as long as your private key remains secret
  - Fewer keys needed
    - $O(n)$ keys
- Disadvantages
  - Slow, often up to 1000x slower than symmetric-key cryptography

# Cryptography and Complexity

- Three classes of complexity:
  - P: solvable in polynomial time, $O(n^c)$
  - NP: nondeterministic solutions in polynomial time, deterministic solutions in exponential time
  - EXP: exponential solutions, $O(c^n)$
- Cryptographic problems should be:
  - Encryption should be P
  - Decryption should be P with key
  - Decryption should be NP for attacker




- Need problems where complexity of solution depends on knowledge of a key

# Modular Arithmetic Review

- Integers modulo prime *p* form an algebraic ring
- Example:
  - Z (mod 7) = {0, 1, 2, 3, 4, 5, 6}
  - Addition: 4 + 5 = 9 = 2 (mod 7)
  - Multiplication: 4 * 5 = 20 = 6 (mod 7)
  - Additive Identity: 4 + 0 = 4 (mod 7)
  - Multiplicative Identity: 4 * 1 = 4 (mod 7)
  - Inverse: 4 * 2 = 8 = 1 (mod 7)
    - $4^{-1} = 2 \pmod 7$
    - $2^{-1} = 4 \pmod 7$
    - Can use Euclidean Algorithm to find inverses (mod p) in polynomial time

# Knapsack Problem

- Finding subset of items that completely fill a knapsack
- Cast mathematically, find a binary selection vector $v_i$ such that: $$\sum_i v_i a_i = T$$
- Vector $a_i$ represents the size of the items and , and T is the total size of the knapsack
- Example:
  - a = {5, 8, 2, 9, 11, 4}
  - T = 14
  - Solution: v = {0, 1, 1, 0, 0, 1}

# Knapsack Problem

- Finding vector v for an arbitrary knapsack is an NP problem
  - Deterministic exponential solution: try every vector $2^n$
  - More efficient: recursive algorithm on sorted knapsack
- Superincreasing knapsack:
  - Special case where $a_n > \sum_{i=1}^{n-1} a_i$
  - Polynomial-time solution exists
- Example:
  - a = {1, 3, 6, 13, 25, 51}
  - T = 32
  - Solution
    - Can't have 51
    - Must have 25, result is 7
    - Can't have 13,
    - Must have 6, result is 1, etc

# Knapsack Problem

- Use knapsack problem for cryptography
  - Plaintext is vector v
  - Ciphertext is target T
  - Key is vector a
- Need two equivalent knapsacks
  - Regular knapsack for encryption, $k_e$ (public key)
  - Superincreasing knapsack for decryption $k_d$ (private key)
  - Need a way to convert a superincreasing knapsack to a regular knapsack
    - Technique: use modular arithmetic
    - $k_e = c\ k_d \pmod{n}$

# Knapsack Problem

- Example:
  - $k_d = \{1, 3, 6, 13, 25, 53\}$
  - $k_e = 51\ k_d \pmod{107} = \{51, 46, 92, 21, 98, 28\}$
  - Message $M = \{0, 1, 1, 0, 1, 1\}$
  - Ciphertext $T = 264$
  - Decrypt using $k_d$
    - Need to “undo” multiplication by 51 (mod 107), use Euclidean algorithm to determine that $51 * 21 \pmod{107} = 1$, so $21 = 51^{-1}$
    - Compute new ciphertext $T' = 264 * 21 \pmod{107} = 87$
    - Must have 53, result is 34
    - Must have 25, result is 9
    - Cannot have 13
    - Must have 6, 3, result is {0, 1, 1, 0, 1, 1}

# Knapsack Problem

- Proposed in 1978 as a public-key encryption scheme
- Analysis in 1983 showed flaws
  - Heuristic techniques for determining multiplier and modulus
  - Results in a polynomial-time algorithm to derive $k_d$ from $k_e$
  - Flaw means that cryptosystems based on transforming a superincreasing knapsack are insecure

# RSA

- Rivest-Shamir-Adleman
- Also introduced in 1978
- Based on the difficulty of factoring a large composite number into two large primes
  - Believed to be an exponential-time problem
  - Polynomial-time algorithms exist for Quantum computers
- Relies on generalization of Fermat's theorem:

$$x^{\varphi(n)} = 1 \pmod{n}$$

- $\varphi(n)$ is the number of numbers less than n, coprime with n
  - Euler's Totient Function
  - For $n = p$, $\varphi(n) = n-1$, for any prime p
  - For $n = pq$, $\varphi(n) = \varphi(p)\,\varphi(q) = (p-1)(q-1)$, for any primes p, q

# RSA

- Uses modular arithmetic, for plaintext P, ciphertext C

$$C = P^e \pmod{n} \qquad P = C^d \pmod{n}$$

- Need values *d*, *e*, *n* to make it work
- Using Fermat's Theorem:
  - Let $n = pq$ for primes $p$, $q$, test for prime is polynomial
  - Let $d = e^{-1} \pmod{\varphi(n)}$, Euclidean algorithm is polynomial
- Then:

$$\begin{aligned} P &= C^d \pmod{n} \\ &= (P^e)^d \pmod{n} \\ &= P^{ed} \pmod{n} \\ &= P^1 \pmod{n} \end{aligned}$$

# RSA

- Direct Attack
  - Attacker needs to be able to compute "Discrete Logarithm"
  - That is, $C = P^e \pmod{n}$
    - If C, e, n known, compute P
    - $\log_P(C) = e \pmod{n}$
    - Solving in R is easy, but in Z (mod $n$) is EXP
- Rather than attack directly, try to find private key
  - Adversary needs to know $\varphi(n)$ to compute $d$ from $e$
  - To know $\varphi(n)$, attacker must know $p$, $q$ used to compute $n$
  - Attack requires factorization

# RSA

- Security of RSA
  - Used in nearly every secure transaction over the Internet
  - Originally *n* was 512 bits (RSA-512)
    - Now crackable in under a year on a standard desktop computer
    - Roughly equivalent to DES
  - Most current Internet sites use RSA-1024
    - Infeasible to crack given current processing power
  - Most new standards and systems recommend RSA-2048
    - RSA-2048 keys are as difficult to crack as AES-128

# El Gamal Encryption

- RSA can be cracked either by:
  - Solving Discrete Logarithm (DL) problem
  - Factoring public key
- Factoring is easier
- Need a cryptosystem that doesn't involve factoring, and based solely on DL problem
- Result would be more secure
  - Shorter key length for the same level of security
- Invented by El Gamal in 1984

# El Gamal Encryption

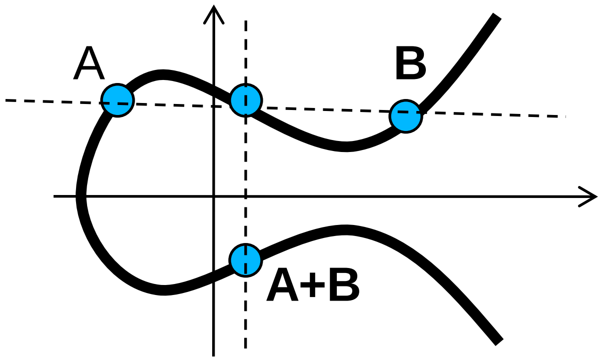
- Use multiplicative group of integers (mod p)
  - Any algebraic group will work
- Key generation
  - Select prime $p$, integers $a$, $x < p$ / private key {x}
  - Compute $r = a^x$ (mod $p$) / public key {p, a, r}
- Encryption
  - Select random integer $y < p$
  - Compute $c_1 = a^y$, $c_2 = Mr^y$ / ciphertext $\{c_1, c_2\}$
- Decryption
  - Compute plaintext = $c_1^{-x} c_2$
  - $c_1^{-x} c_2 = (a^y)^{-x} Mr^y = M a^{-xy} (a^x)^y = M a^{xy} a^{-xy} = M$ (mod $p$)

# El Gamal Encryption

- Basic security provided by the discrete logarithm problem
- Other attacks: security actually limited
  - Computational Diffie-Hellman problem
  - Decisional Diffie-Hellman problem
  - Will discuss these in detail next week
- System is malleable
  - Example: adversary can change $c_2'=2c_2$
  - Adversary decrypts $c_1^{-x}c_2' = 2M$
  - Deterministic change to ciphertext yields deterministic change in plaintext
  - Still need integrity protection

# Elliptic Curve Cryptography

- Elliptic curves can be used to create an algebraic group
- Combined with El Gamal Encryption to perform Elliptic Curve Cryptography
- Basic idea:
  - Points on a curve are group elements
  - Can be “added together” by:
    - Find third point colinear with first two
    - Reflect across axis
  - Efficient algorithm exists for computation
  - Exponentiation: Compute *c* * A, where *c* is an integer constant, as *c* * A = A+A+A+…+A (*c* times)
  - Forms an algebraic group with difficult discrete logarithm problem



# Elliptic Curve Cryptography

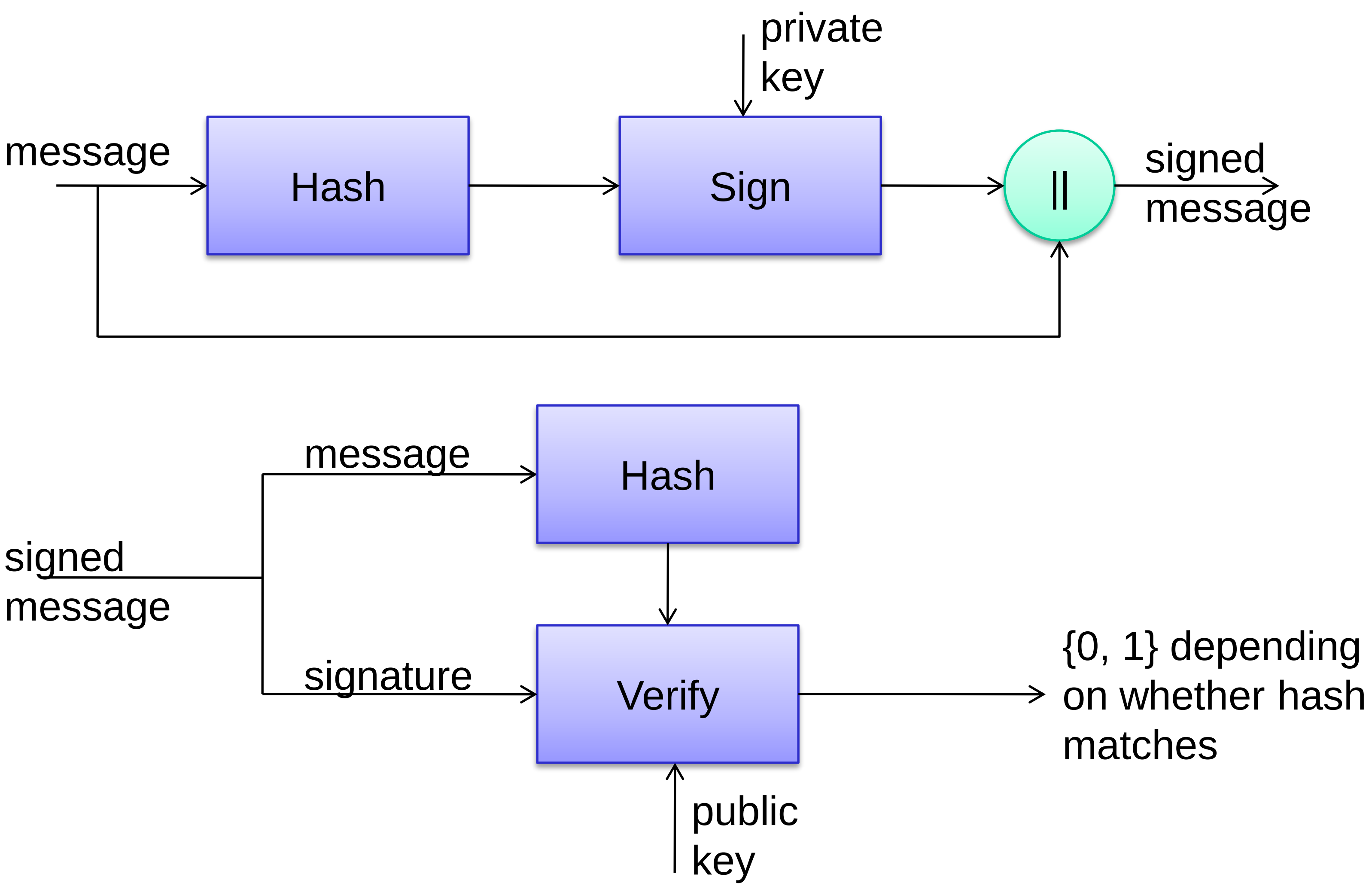
- Advantages
  - Security bounded by DL problem rather than factoring problem
  - Can use significantly shorter key sizes
  - ECC-160 roughly equivalent to RSA-1024
    - MUCH shorter key sizes, better for storage, transmission
  - Still secure even if someone finds polynomial time for factoring integers
  - As RSA keys get longer, equivalently-secure ECC is more efficient in both hardware and software
- Disadvantages
  - Less institutionalized, most certificates don’t support it
- Future of ECC
  - Patents by Certicom discourage use, expiring soon
  - USG pushing for use within USG systems

# Digital Signatures

- Before, MIC provided message integrity
- Need a public-key equivalent
- Basic approach:
  - Most public-key systems have interchangable keys
    - RSA: could use either *d* or *e* to encrypt or decrypt, one undoes the other
  - Compute a hash of the message, and “encrypt” it with the private key
  - Recipient “decrypts” with the public key, verifies the hash

# Digital Signatures

- Overall Architecture:



# RSA Digital Signatures

- As mentioned before: simply “encrypt” with the private key
  - M = Message, S = Signature
  - To sign: $S = Hash(M)^d \pmod{n}$
  - To verify, see if $Hash(M) = S^e \pmod{n}$
- Relies on security of hash function
  - If a collision can be found, an attacker can change M to M’ such that Hash(M)=Hash(M’)
  - Same signature S would be valid for both M and M’

# Digital Signature Algorithm (DSA)

- DSA is NIST standard for digital signatures
- Based on El Gamal signature scheme
  - Similar to El Gamal Encryption
  - Relies on DL problem rather than factorization
- Key Generation:
  - Select prime $p$, integers $a$, $x < p$ / private key = {x}
  - Compute $y = a^x \pmod{p}$ / public key = {p, a, y}
- Signature:
  - Select random integer $k < p-1$
  - Compute $r = a^k \pmod{p}$
  - Compute $s = k^{-1}(\text{Hash}(M) - xr) \pmod{p-1}$
  - Signature: $\{r, s\}$
- Verify:
  - Compute $v = y^r r^s \pmod{p}$
  - Determine if $v = a^{\text{Hash}(M)} \pmod{p}$

$$
\begin{aligned}
y^r r^s &= (a^x)^r r^{\left(k^{-1}(Hash(M)-xr)\right)} \pmod{p} \\
&= a^{xr}(a^k)^{\left(k^{-1}(Hash(M)-xr)\right)} \pmod{p} \\
&= a^{\left(xr+kk^{-1}(Hash(M)-xr)\right)} \pmod{p} \\
&= a^{xr+Hash(M)-xr} \pmod{p} \\
&= a^{Hash(M)} \pmod{p}
\end{aligned}
$$

# Digital Signature Algorithm

- DSA can also be used with Ellipitic Curve Group rather than multiplicative integers
  - Called ECDSA
  - Again requires shorter key for equivalent security
  - Based on El Gamal Signatures
- Most digital signature systems use DSA rather than RSA signatures
- Very few use ECDSA

# Quantum Cryptography

- Drastically different than mathematical cryptography explored so far
- Encodes data as photons of light
- Photons can spin in different orientations: ⇨⇧⬁⬀
- Polarized filters can detect photons
  - + filter: detects ⇨⇧ correctly, ⬁⬀ randomly
  - X filter: detects ⬁⬀ correctly, ⇨⇧ randomly
- Sender assigns 0 to ⇨ or ⬁, 1 to ⇧ or ⬀
- Sender's message {0, 1, 1, 0, 1} to {⬁, ⇧, ⬀, ⬁, ⇧}
- Receiver uses random filters to detect {+, +, X, X, X}
- Receiver detects {1, 1, 1, 0, 1} (first, last filters incorrect)
- Receiver sends filter list to sender, sender indicates which were correct
- Receiver now correctly knows {?, 1, 1, 0, ?}
- Use error-correcting code to communicate over channel

# Quantum Cryptography

- Security is based on the Heisenberg Uncertainty Principle
  - If you measure the rotation of a photon, you randomly change the rotation
  - Sender/Receiver could detect statistically abnormal error rate in the channel
- Implementation issues
  - Currently difficult to send exactly one photon of light
  - Approaches use a laser and attenuate the output such that statistically the expected number of photons is 1 per bit
- Applications
  - Doesn't rely on DL or factorization, therefore immune to Quantum Cryptanalysis; may be one of the only viable cryptosystems
  - Currently geared toward satellite communications

# Public-Key Cryptosystems (PKCS)

- PKCS encapsulations
  - RSA has defined proprietary encapsulations of data into encrypted, signed blobs
  - PKCS #1, #2, etc, defined different encodings
  - Some offer encryption, others signatures, or both
- Transaction Layer Security (TLS)
  - Fundamental basis of secure communications over the Internet
  - Uses RSA, etc, for key agreement (discuss in detail next week)
- Email standards
  - CMS (Cryptographic Message Syntax) used for SMIME, use RSA/DSS
  - PGP and GPG are commonly used for email encryption, use El Gamal

# Public Key Infrastructure

- ANSI X.509 standards
  - Define how to format public keys for exchange over networks
  - Major use: definition of certificate format
- Certificates are public keys signed by an external authority
  - e.g. Verisign
  - Trusted third party, called Certificate Authority (CA)
- Prevents MITM attacks
  - Someone sends you a public key to communicate with them securely
  - How do you know it's really the public key of the person you want to communicate with?
  - Have a trusted third party sign the key as actually being owned by someone
  - Anyone can create a CA, but popular software applications only list major companies, others have to be added manually