Polyhedral Combinatorics and Combinatorial Optimization

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1 Introduction

Combinatorial optimization searches for an optimum object in a finite collection of objects. Typically, the collection has a concise representation (like a graph), while the number of objects is huge — more precisely, grows exponentially in the size of the representation (like all matchings or all Hamiltonian circuits). So scanning all objects one by one and selecting the best one is not an option. More efficient methods should be found.

In the 1960s, Edmonds advocated the idea to call a method efficient if its running time is bounded by a polynomial in the size of the representation. Since then, this criterion has won broad acceptance, also because Edmonds found polynomial-time algorithms for several important combinatorial optimization problems (like the matching problem). The class of polynomial-time solvable problems is denoted by P.

Further relief in the landscape of combinatorial optimization was discovered around 1970 when Cook and Karp found out that several other prominent combinatorial optimization problems (including the traveling salesman problem) are the hardest in a large natural class of problems, the class NP. The class NP includes most combinatorial optimization problems. Any problem in NP can be reduced to such 'NP-complete' problems. All NP-complete problems are equivalent in the sense that the polynomial-time solvability of one of them implies the same for all of them.

Almost every combinatorial optimization problem has since been either proved to be polynomial-time solvable or NP-complete — and none of the problems have been proved to be both. This spotlights the big mystery: are the two properties disjoint (equivalently, P≠NP), or do they coincide (P=NP)?

Polyhedral and linear programming techniques have turned out to be essential in solving combinatorial optimization problems and studying their complexity. Often a polynomial-time algorithm yields, as a by-product, a description (in terms of inequalities) of an associated polyhedron. Conversely, an appropriate description of the polyhedron often implies the polynomial-time solvability of the

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associated optimization problem, by applying linear programming techniques. With the duality theorem of linear programming, polyhedral characterizations yield min-max relations, and vice versa. This area of discrete mathematics is called *polyhedral combinatorics*. We give some basic, illustrative examples. For an extensive survey, we refer to Schrijver [39]. Background on linear programming can be found in [38].

2 Perfect matchings

Let $G = (V, E)$ be an undirected graph. A *perfect matching* in G is a set M of disjoint edges covering all vertices. Let $w : E \to \mathbb{R}_+$. For any perfect matching M, denote

(1) $$w(M) := \sum_{e \in M} w(e).$$

We will call $w(M)$ the *weight* of M.

Suppose now that we want to find a perfect matching M in G with weight $w(M)$ as small as possible. In notation, we want to 'solve'

(2) $$\min\{w(M) \mid M \text{ perfect matching in } G\}.$$

This problem shows up in several practical situation, for instance when an optimum assignment or schedule has to be determined.

We can formulate problem (2) equivalently as follows. For any perfect matching M, denote the incidence vector of M in $\mathbb{R}^E$ by χ^M; that is,

(3) $$\chi^M(e) := \begin{cases} 1 & \text{if } e \in M, \\ 0 & \text{if } e \notin M, \end{cases}$$

for $e \in E$. Considering w as a *vector* in $\mathbb{R}^E$, we have $w(M) = w^\mathsf{T}\chi^M$. Hence problem (2) can be rewritten as

(4) $$\min\{w^\mathsf{T}\chi^M \mid M \text{ perfect matching in } G\}.$$

This amounts to minimizing the linear function $w^\mathsf{T}x$ over a finite set of vectors. Therefore, the optimum value does not change if we minimize over the *convex hull* of these vectors:

(5) $$\min\{w^\mathsf{T}x \mid x \in \text{conv.hull}\{\chi^M \mid M \text{ perfect matching in } G\}\}.$$

The set

(6) $\quad\quad \text{conv.hull}\{\chi^M \mid M \text{ perfect matching in } G\}$

is a polytope in $\mathbb{R}^E$, called the *perfect matching polytope* of G.

As it is a polytope, there exist a matrix A and a vector b such that

(7) $\quad\quad \text{conv.hull}\{\chi^M \mid M \text{ perfect matching in } G\} = \{x \in \mathbb{R}^E \mid Ax \leq b\}.$

Then problem (5) is equivalent to

(8) $\quad\quad \min\{w^{\mathsf{T}}x \mid Ax \leq b\}.$

In this way we have formulated the original combinatorial problem (2) as a *linear programming* problem. This enables us to apply linear programming methods to study the original problem.

The question at this point is, however, how to find the matrix A and the vector b. We know that A and b do exist, but we must know them in order to apply linear programming methods.

For *bipartite* graphs, such an A and b can easily be found. (A graph is *bipartite* if its vertices can be split into two classes such that each edge connects a vertex in one class with a vertex in the other class.) If G is bipartite, the matching polytope of G is equal to the set of all vectors $x \in \mathbb{R}^E$ satisfying

(9)
$$\begin{aligned} x(e) &\geq 0 && \text{for } e \in E, \\ \sum_{e \ni v} x(e) &= 1 && \text{for } v \in V. \end{aligned}$$

(The sum ranges over all edges e containing v.)

This is in fact equivalent to a theorem of Birkhoff [2], saying that each doubly stochastic matrix is a convex combination of permutation matrices. (A matrix is *doubly stochastic* if it is nonnegative and each row sum and each column sum is equal to 1. A *permutation matrix* is a $0,1$ matrix with precisely one 1 in each row and each column.)

It is not difficult to show that the perfect matching polytope for bipartite graphs is indeed completely determined by (9). First note that the perfect matching polytope is contained in the polytope determined by (9), since χ^M satisfies (9) for each perfect matching M. To see the reverse inclusion, we note that, if G is bipartite, then the $V \times E$ incidence matrix A_G of G is *totally unimodular*, i.e., each square submatrix has determinant belonging to $\{0, +1, -1\}$. (This was shown by Poincaré [37].)

Theorem 1. *The incidence matrix A_G of a bipartite graph $G = (V, E)$ is totally unimodular.*

Proof. Let B be a square submatrix of A_G, of order k say. We show that $\det B$ equals 0 or ± 1 by induction on t. If $k = 1$, the statement is trivial. So let $k > 1$.

We distinguish three cases.

Case 1: B has a column with only 0's. Then $\det B{=}0$.

Case 2: B has a column with exactly one 1. In that case we can write (possibly after permuting rows or columns):

$$(10) \qquad B = \begin{pmatrix} 1 & b^{\mathsf{T}} \\ \mathbf{0} & B' \end{pmatrix},$$

for some matrix B' and vector b, where $\mathbf{0}$ denotes the all-zero vector in $\mathbb{R}^{t-1}$. By the induction hypothesis, $\det B' \in \{0, \pm 1\}$. Hence, by (10), $\det B \in \{0, \pm 1\}$.

Case 3. Each column of B contains exactly two 1's. Then, since G is bipartite, we can write (possibly after permuting rows):

$$(11) \qquad B = \begin{pmatrix} B' \\ B'' \end{pmatrix},$$

in such a way that each column of B' contains exactly one 1 and each column of B'' contains exactly one 1. So adding up all rows in B' gives the all-one vector, and also adding up all rows in B'' gives the all-one vector. The rows of B therefore are linearly dependent, and hence $\det B{=}0$. ∎

The total unimodularity of A_G implies that the vertices of the polytope determined by (9) are *integer* vectors, i.e., belong to $\mathbb{Z}^E$. Now each integer vector satisfying (9) must trivially be equal to χ^M for some perfect matching M. Hence,

(12) if G is bipartite, the perfect matching polytope is determined by (9).

We therefore can apply linear programming techniques to handle problem (2). Thus we can find a minimum-weight perfect matching in a bipartite graph in polynomial time, with any polynomial-time linear programming algorithm. Moreover, the duality theorem of linear programming gives

$$(13) \qquad \begin{aligned} &\min\{w(M) \mid M \text{ perfect matching in } G\} \\ &= \min\{w^{\mathsf{T}}x \mid x \geq \mathbf{0}, A_G x = \mathbf{1}\} \\ &= \max\{y^{\mathsf{T}}\mathbf{1} \mid y \in \mathbb{R}^V, y^{\mathsf{T}} A_G \geq w^{\mathsf{T}}\}. \end{aligned}$$

($\mathbf{1}$ denotes an all-one vector.) This is an example of a *min-max formula* that can be derived from a polyhedral characterization. Conversely, min-max formulas (in particular in a weighted form) often give polyhedral characterizations.

The polyhedral description together with linear programming duality also gives a *certificate* of optimality of a perfect matching M: to convince your 'boss' that a certain perfect matching M has minimum weight, it is possible and sufficient to display a vector y in $\mathbb{R}^V$ satisfying $y^{\mathsf{T}} A_G \geq w^{\mathsf{T}}$ and $y^T\mathbf{1} = w(M)$.

In other words, it yields a *good characterization* for the minimum-weight perfect matching problem in bipartite graphs.

3 But what about nonbipartite graphs?

For general, nonbipartite graphs G, the perfect matching polytope is not determined by (9). For instance, if G is an odd circuit, then the vector $x \in \mathbb{R}^E$ defined by $x(e) := \frac{1}{2}$ for all $e \in E$, satisfies (9) but does not belong to the perfect matching polytope of G (as G has no perfect matching at all).

A pioneering and central theorem in polyhedral combinatorics of Edmonds [8] gives a complete description of the inequalities needed to describe the perfect matching polytope for arbitrary graphs: one should add to (9) the inequalities

$$\text{(14)} \qquad \sum_{e \in \delta(U)} x(e) \geq 1 \text{ for each odd-size subset } U \text{ of } V.$$

Here $\delta(U)$ denotes the set of edges connecting U and $V \setminus U$.

Trivially, the incidence vector χ^M of any perfect matching M satisfies (14). So the perfect matching polytope of G is contained in the polytope determined by (9) and (14). The content of Edmonds' theorem is the converse inclusion.

Theorem 2. *For any graph G, the perfect matching polytope is determined by* (9) *and* (14).

Proof. Clearly, the perfect matching polytope is contained in the polytope Q determined by (9) and (14). Suppose that the converse inclusion does not hold. Then we can choose a vertex x of Q that is not in the perfect matching polytope.

We may assume that we have chosen this counterexample such that $|V|+|E|$ is as small as possible. Hence $0 < x(e) < 1$ for all $e \in E$ (otherwise, if $x(e) = 0$, we can delete e, and if $x(e) = 1$, we can delete e and its ends). So each degree of G is at least 2, and hence $|E| \geq |V|$. If $|E| = |V|$, each degree is 2, in which case the theorem is trivially true. So $|E| > |V|$. Note also that $|V|$ is even, since otherwise $Q = \emptyset$ (consider $U := V$ in (14)).

As x is a vertex of Q, there exist $|E|$ linearly independent constraints among (9) and (14) satisfied with equality. Since $|E| > |V|$, there is an odd subset U of V with $3 \leq |U| \leq |V| - 3$ and $\sum_{e \in \delta(U)} x(e) = 1$.

Consider the projections x' and x'' of x to the edge sets of the graphs $G/\overline{U}$ and G/U, respectively (where $\overline{U} := V \setminus U$, and where G/W is the graph obtained from G by contracting all vertices in W to one vertex). Here we keep parallel edges.

Then x' and x'' satisfy (9) and (14) for $G/\overline{U}$ and G/U, respectively, and hence belong to the perfect matching polytopes of $G/\overline{U}$ and G/U, by the minimality of $|V| + |E|$.

So there is a k such that $G/\overline{U}$ has perfect matchings $M_1', \ldots, M_k'$ and G/U has perfect matchings $M_1'', \ldots, M_k''$ with

$$x' = \frac{1}{k}\sum_{i=1}^{k}\chi^{M_i'} \text{ and } x'' = \frac{1}{k}\sum_{i=1}^{k}\chi^{M_i''}. \tag{15}$$

(Note that x is rational as it is a vertex of Q.)

Now for each $e \in \delta(U)$, the number of i with $e \in M_i'$ is equal to $kx'(e) = kx(e) = kx''(e)$, which is equal to the number of i with $e \in M_i''$. Hence we can assume that, for each $i = 1, \ldots, k$, M_i' and M_i'' have an edge in $\delta(U)$ in common. So $M_i := M_i' \cup M_i''$ is a perfect matching of G. Then

$$x = \frac{1}{k}\sum_{i=1}^{k}\chi^{M_i}. \tag{16}$$

Hence x belongs to the perfect matching polytope of G. ∎

In fact, Edmonds designed a polynomial-time algorithm to find a minimum-weight perfect matching in a graph, which gave this polyhedral characterization as a by-product. Conversely, from the characterization one may derive the polynomial-time solvability of the weighted perfect matching problem. In applying linear programming methods for this, one will be faced with the fact that (9),(14) consists of exponentially many inequalities, since there exist exponentially many odd-size subsets U of V. So in order to solve the problem with linear programming methods, we cannot just list all inequalities.

However, the *ellipsoid method* for linear programming (Khachiyan [26]) does not require that all inequalities are listed a priori ([22,23]). It suffices to have a polynomial-time algorithm answering the question:

(17) given $x \in \mathbb{R}^E$, does x belong to the perfect matching polytope of G?

Such an algorithm indeed exists, as it has been shown that the inequalities (9) and (14) can be checked in time bounded by a polynomial in $|V|$, $|E|$, and the size of x. This method obviously should avoid testing all inequalities (14) one by one.

Combining the description of the perfect matching polytope with the duality theorem of linear programming gives a min-max formula for the minimum weight of a perfect matching. It again yields a certificate of optimality: if we have a perfect matching M, we can convince our 'boss' that M has minimum weight, by supplying a dual solution y of objective value $w(M)$. So the minimum-weight perfect matching problem has a good characterization — i.e., belongs to NP∩co-NP.

This gives one motivation for studying polyhedral methods. The ellipsoid method proves polynomial-time solvability, it however does not yield a practical method, but rather an incentive to search for a practically efficient algorithm. The polyhedral method can be helpful also in this, e.g., by imitating the simplex method with a constraint generation technique, or by a primal-dual approach.

We note that Edmonds' theorem is equivalent to the following.

(18) The convex hull of the *symmetric* permutation matrices in $\mathbb{R}^{n\times n}$ is equal to the set of doubly stochastic matrices with the property that for each odd number k and each principal submatrix B of order k, the sum of the entries in B is at most $k-1$.

4 Hamiltonian circuits and the traveling salesman problem

As we saw, perfect matchings form an area where the search for an inequality system determining the corresponding polytope has been successful. This is in contrast with, for instance, Hamiltonian circuits. (A *Hamiltonian circuit* is a circuit covering all vertices.) No full description in terms of inequalities of the convex hull of the incidence vectors of edge sets of Hamiltonian circuits — the *traveling salesman polytope* — is known. The corresponding optimization problem is the traveling salesman problem: 'find a Hamiltonian circuit of minimum weight', which problem is NP-complete.

This implies that, unless NP=co-NP, there exist facet-inducing inequalities for the traveling salesman polytope that have no polynomial-time certificate of validity. Otherwise, linear programming duality would yield a good characterization. So unless NP=co-NP there is no hope for an appropriate characterization of the traveling salesman polytope. It can be seen that the following 'obvious' set of inequalities is not enough to determine the traveling salesman polytope:

(19)
$$\begin{aligned} & x(e) \geq 0 && \text{for } e \in E,\\ & \sum_{e \ni v} x(e) = 2 && \text{for } v \in V,\\ & \sum_{e \in \delta(U)} x(e) \geq 2 && \text{for } \emptyset \neq U \neq V. \end{aligned}$$

In matrix terms, unless NP=co-NP, there is no hope for an appropriate description of the convex hull of those $n \times n$ permutation matrices made by a permuation with precisely one orbit. (Simply requiring that the entries in any nonempty proper principal submatrix of order k add up to at most $k-1$ is not enough.)

Moreover, unless NP=P, there is no polynomial-time algorithm answering the question

(20) given $x \in \mathbb{R}^E$, does x belong to the traveling salesman polytope?

Otherwise, the ellipsoid method would give the polynomial-time solvability of the traveling salesman problem.

Nevertheless, polyhedral combinatorics can be applied to the traveling salesman problem in a positive way. If we include the traveling salesman polytope in a larger polytope (a *relaxation*) over which we *can* optimize in polynomial time (which is the case for the polytope determined by (19)), we obtain a polynomial-time computable bound for the traveling salesman problem. The closer the relaxation is to the traveling salesman polytope, the better the bound is. This can be very useful in a branch-and-bound algorithm. This idea originates from Dantzig, Fulkerson, and Johnson [6].

5 Stable sets and semidefinite programming

Related to the problems described above is the problem of finding a maximum-size stable set in a graph $G = (V, E)$ (and more generally, a maximum-weight stable set, but we will restrict ourselves here to the cardinality case). Here a subset S of V is called *stable* if any two vertices in S are nonadjacent in G. The problem comes up in practice for instance when assigning frquencies to radio stations or mobile phones.

Finding a maximum-size stable set is again an NP-complete problem, so no good description of the corresponding *stable set polytope* (the convex hull of the incidence vectors in $\mathbb{R}^V$ of the stable sets) may be expected.

However, for certain graphs, the so-called *perfect graphs*, a maximum-size stable set can be found in polynomial time ([22]). The basic idea is to apply semidefinite programming to calculate the following bound of Lovász [34] on the maximum size $\alpha(G)$ of a stable set in G:

(21) $$\vartheta(G) := \max\{\mathbf{1}^{\mathsf{T}} M \mathbf{1} \mid M \in \mathbb{R}^{n\times n} \text{ positive semidefinite, } M_{i,j} = 0 \text{ if } ij \in E, \ \mathrm{trace}(M) = 1\}.$$

Here we assume without loss of generality that G has vertex set $\{1, \ldots, n\}$.

To see that $\alpha(G) \leq \vartheta(G)$, let S be a maximum-size stable set in G, and define the matrix M by

(22) $$M := |S|^{-1} \cdot \chi^S (\chi^S)^{\mathsf{T}},$$

where χ^S is the incidence vector of S in $\mathbb{R}^V$, taken as *column vector*. Then M is positive semidefinite, $M_{i,j} = 0$ if $ij \in E$, and trace$M = 1$. So $\alpha(G) = |S| = \mathbf{1}^{\mathsf{T}} M \mathbf{1} \leq \vartheta(G)$.

The value $\vartheta(G)$ can be calculated in polynomial time, as it is a *semidefinite programming problem*. A generic form of such a problem is: given $c_1, \ldots, c_t \in \mathbb{R}$ and real symmetric matrices $A_0, \ldots, A_t, B$ (of equal dimensions), find $x_1, \ldots, x_t \in \mathbb{R}$ that maximize $\sum_i c_i x_i$ subject to the condition that $(\sum_i x_i A_i) - B$ is positive semidefinite. If all A_i and B are diagonal matrices, we have a linear programming problem. Semidefinite programming problems can be solved in polynomial time, with the ellipsoid method or with an 'interior-point method'.

It follows that if $\alpha(G) = \vartheta(G)$, we can calculate $\alpha(G)$ in polynomial time. Moreover, if

(23) $\alpha(G') = \vartheta(G')$ for each induced subgraph G' of G,

then we can find a maximum-size stable set in polynomial time. (An *induced subgraph* is a subgraph (V', E') of (V, E) with $V' \subseteq V$ and $E' = \{ij \in E \mid i, j \in V'\}$.)

Which graphs satisfy (23)? First, it was shown by Lovász [35] that these are precisely the graphs G such that

(24) $\alpha(G') = \gamma(\overline{G'})$ for each induced subgraph G' of G.

Here $\gamma(H)$ denotes the colouring number of H, and $\overline{H}$ denotes the complementary graph of H (whose edges are precisely the nonedges of H).

Berge [1] introduced the name *perfect* for graphs G satisfying (24), and he conjectured that these are precisely those graphs G with the property that neither G nor $\overline{G}$ contains a chordless circuit of odd length ≥ 5. (Necessity of this condition is easy.) This *strong perfect graph conjecture* was only recently proved by Chudnovsky, Robertson, Seymour, and Thomas [3], requiring deep decomposition techniques for graphs. Berge's *weak perfect graph conjecture*, stating that the complement $\overline{G}$ of any perfect graph is perfect again, was shown earlier by Lovász [33].

6 Historically

The first min-max relations in combinatorial optimization were proved by Dénes Kőnig [27,28], on edge-colouring and matchings in bipartite graphs, and by Karl Menger [36], on disjoint paths in graphs. The matching theorem of Kőnig was extended to the weighted case by Egerváry [14]. The proofs by Kőnig and Egerváry were in principal algorithmic, and also for Menger's theorem an algorithmic proof was given in the 1930s. The theorem of Egerváry may be seen as polyhedral.

Applying linear programming techniques to combinatorial optimization problems came along with the introduction of linear programming in the 1940s and 1950s. In fact, linear programming forms the hinge in the history of combinatorial optimization. Its initial conception by Kantorovich [25] and Koopmans [29] was motivated by combinatorial applications, in particular in transportation and transshipment.

After the formulation of linear programming as generic problem, and the development in 1947 by Dantzig [5] of the simplex method as a tool, one has tried to attack about all combinatorial optimization problems with linear programming techniques, quite often very successfully. In the 1950s, Dantzig [4], Ford and Fulkerson [16,15,17], Hoffman [24], Kuhn [30,31], and others studied problems like the transportation, maximum flow, and assignment problems. These problems can be reduced to linear programming by the total unimodularity of the underlying matrix, thus yielding extensions and polyhedral and algorithmic interpretations of the earlier results of Kőnig, Egerváry, and Menger. Kuhn realized that the polyhedral methods of Egerváry for weighted bipartite matching are in fact algorithmic, and yield the efficient 'Hungarian' method for the assignment problem. Dantzig, Fulkerson, and Johnson [6,7] gave a solution method for the traveling salesman problem, based on linear programming with a rudimentary, combinatorial version of a cutting plane technique.

A considerable extension and deepening, and a major justification, of the field of polyhedral combinatorics was obtained in the 1960s and 1970s by the work and pioneering vision of Edmonds [8,9,10,11,12,13]. He characterized basic polytopes like the perfect matching polytope, the arborescence polytope, and the matroid intersection polytope; he introduced (with Giles) the important concept of total dual integrality; and he advocated the interconnections between polyhedra, min-max relations, good characterizations, and efficient algorithms.

Also during the 1960s and 1970s, Fulkerson [18,19,20,21] designed the clarifying framework of blocking and antiblocking polyhedra, throwing new light by the classical polarity of vertices and facets of polyhedra on combinatorial min-max relations and enabling, with a theorem of Lehman [32], the deduction of one polyhedral characterization from another. It stood at the basis of the solution of Berge's weak perfect graph conjecture in 1972 by Lovász [33]. As mentioned, Berge's *strong perfect graph conjecture* was recently proved by Chudnovsky, Robertson, Seymour, and Thomas [3].

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