CONTRIBUTIONS TO THE THEORY OF GAMES

VOLUME II

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A CERTAIN ZERO-SUM TWO-PERSON GAME EQUIVALENT TO THE OPTIMAL ASSIGNMENT PROBLEM[1]

John von Neumann

The optimal assignment problem is as follows: given n persons and n jobs, and a set of real numbers a_{ij}, each representing the value of the i^{th} person in the j^{th} job, what assignments of persons to jobs will yield maximum total value? A solution can be expressed as a permutation of n objects, or, equivalently, as an $n \times n$ permutation matrix. (Such a matrix can be expressed by $\delta_{i^P, j}$, where δ_{ij} is the Kronecker symbol and i^P is the image of i under permutation P.) The value of a particular assignment (i.e., permutation) P will be $\sum_i a_{i,i^P}$. Without further investigation, a direct solution of the problem appears to require $n!$ steps, -- the testing of each permutation to find the optimal permutations giving the maximum $\sum_i a_{i,i^P}$.

We observe that the solution is invariant under the matrix transformation $a_{ij} \longrightarrow a_{ij} + u_i + v_j$, where u_i and v_j are any sets of constants. It is clear that $\sum_i u_i + \sum_j v_j$ will be added to each assignment value, and that thus the order of values, particularly the maxima, will be preserved. This enables us to transform a given assignment problem with possibly negative a_{ij} to an equivalent one with strictly positive a_{ij}, by adding large enough positive u_i and v_j.

We shall now construct a certain related 2-person game and we shall show that the extreme optimal strategies can be expressed in terms of the optimal permutation matrices in the assignment problem. (The game matrix for this game will be $2n \times n^2$. From this it is not difficult to infer how many steps are needed to get significant approximate solutions with the method of G. W. Brown and J. von Neumann. [Cf.: "Contributions to the Theory of Games," Annals of Mathematics Studies, No. 24, Princeton University Press, 1950 -- pp. 73-79, especially § 5.] It turns out that this number is a moderate power of n, i.e., considerably smaller than the "obvious" estimate $n!$ mentioned earlier.)

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[1] Editors' Note: This is a transcript, prepared under Office of Naval Research sponsorship by Hartley Rogers, Jr., of a seminar talk given by Professor von Neumann at Princeton University, October 26, 1951.

We first construct a simple preliminary game, the 1-dimensional game: We may think of the game as played with a set of n cells or boxes indexed $i = 1, \ldots, n$.

Move 1: Player I 'hides' in a cell.

Move 2: Player II, ignorant of I's choice, attempts to 'find' player I by similarly choosing a cell.

This is a play. The payoff is determined by a set of α_i (positive). If player I is 'found' in cell i, he pays player II the amount α_i; otherwise he pays 0.

What are the optimal strategies for I? Let his strategy be to choose cell i with probability x_i. Then player II will obtain expected payoff $\alpha_i x_i$ by choosing i. Hence he will choose a cell i for which $\alpha_i x_i$ is maximum. The value for him will thus be $\max_i (\alpha_i x_i)$.

Now let $x = (x_i)$ be optimal for I. Assume that an $\alpha_j x_j < \max_i (\alpha_i x_i)$. Choose $\varepsilon > 0$ such that $\alpha_j (x_j + \varepsilon) = \max_i (\alpha_i x_i)$. Define

$$x_i' \begin{cases} = x_j + \varepsilon & \text{for } i = j \\ = x_i & \text{otherwise} \end{cases} .$$

Then $\max_i (\alpha_i x_i') = \max_i (\alpha_i x_i)$, and $\sum_i x_i' = \sum_i x_i + \varepsilon = 1 + \varepsilon$. Hence the

$$x_i'' = \frac{x_i'}{1 + \varepsilon}$$

can be used as probabilities, and

$$\max_i (\alpha_i x_i'') = \frac{\max_i (\alpha_i x_i')}{1 + \varepsilon} < \max_i (\alpha_i x_i) ,$$

i.e., $x = (x_i)$ was not optimal. Thus necessarily all $\alpha_j x_j = \max_i (\alpha_i x_i)$, i.e., $\alpha_1 x_1 = \ldots = \alpha_n x_n = A$. Now $\sum_i x_i = 1$ implies $A = 1/\sum_i \frac{1}{\alpha_i}$, and, of course, $x_i = \frac{A}{\alpha_i}$. The value of the game (for II) is clearly A.

We now introduce the game in which our particular interest lies. We call it the 2-dimensional game; it is a generalization of the 1-dimensional game as follows:

The cells are doubly indexed from 1 to n. (They may be thought of as fields in an $n \times n$ matrix.)

Move 1: Player I hides as before.

Move 2: Player II now attempts to 'find' I by guessing either of the indices of the cell in which player I has hidden.

He must state which index he is guessing. (I.e., II attempts to pick the row, or the column, of I.) Player I, if so 'found' in cell i,j pays to

II the amount α_{ij}, where the α_{ij} are a given set of positive numbers; otherwise he pays 0.

Thus player I has n^2 pure strategies and player II has $2n$.

We now discuss optimal strategies for player I. Let his mixed strategy be $x = (x_{ij})$, $\sum_{ij} x_{ij} = 1$, where each x_{ij} represents the probability of his hiding in cell i,j. Then player II's pure strategies will give a return of $\sum_j \alpha_{ij} x_{ij}$ for row choice i, or $\sum_i \alpha_{ij} x_{ij}$ for column choice j. As in the 1-dimensional game, he can now simply play pure strategies giving the maximum such return. Player I will try to choose x minimising this return. Thus the value of the game (for II) will be:

$$\underset{x}{\text{Min}}\ \underset{i',j'}{\text{Max}} \left(\sum_j \alpha_{i'j} x_{i'j} , \sum_i \alpha_{ij'} x_{ij'} \right) .$$

The characterization of I's strategies is not quite as easy as before. The simple direct compensatory adjustment of the 1-dimensional game cannot be made.

For further progress, we obtain certain results on the geometry of convex bodies.

We define:

R = Set of all vectors $z = (z_{ij})$ (in n^2 dimensions), such that

$$z_{ij} \geqq 0, \sum_j z_{ij} = 1 , \sum_i z_{ij} = 1 .$$

S = Set of all vectors $z = (z_{ij})$ (in n^2 dimensions), such that

$$z_{ij} \geqq 0, \sum_j z_{ij} \leqq 1 , \sum_i z_{ij} \leqq 1 .$$

T = Set of all vectors $z = (z_{ij})$ (in n^2 dimensions), such that $z_{ij} = \delta_{i^P, j}$ for some permutation P of the integers $1, \ldots, n$. (T thus consists of the $n \times n$ permutation matrices.)

We prove two lemmas:

LEMMA 1. S = Set of all z such that $z \leqq$ some $w \in R$.

PROOF. $S \supseteq$ this set is immediate.

$S \subseteq$ this set is shown as follows: For any $z \in S$ let $N(z)$ be the number of all i with $\sum_j z_{ij} \lneqq 1$ plus the number of all j with $\sum_i z_{ij} \lneqq 1$. Clearly $N(z) = 0, 1, 2, \ldots$ and R is the set of those $z \in S$ for which $N(z) = 0$.

Now consider a $z \in S$. If $z \notin R$, then there is either $\sum_j z_{ij} < 1$ for some i, or $\sum_j z_{ij} < 1$ for some j. However, all $\sum_j z_{ij} \leqq 1$, all $\sum_i z_{ij} \leqq 1$, and $\sum_i(\sum_j z_{ij}) = \sum_j(\sum_i z_{ij})$. Hence $\sum_i z_{ij} < 1$ for some i implies $\sum_i z_{ij} < 1$ for some j, and conversely. Therefore both $\sum_j z_{ij} < 1$ for some i, say $i = i'$, and $\sum_i z_{ij} < 1$ for some j, say $j = j'$. Choose

$$\varepsilon = 1 - \mathrm{Max}\left(\sum_j x_{i'j}, \sum_i x_{ij'}\right).$$

Then $\varepsilon > 0$. Define

$$z'_{ij} \begin{cases} = z_{i'j'} + \varepsilon & \text{for } i = i', j = j' \\ = z_{ij} & \text{otherwise} \end{cases}.$$

Then $z' = (z'_{ij}) \in S$, also always $z_{ij} \leqq z'_{ij}$, i.e., $z \leqq z'$; and either $\sum_j z'_{i'j} = 1$ or $\sum_i z'_{i'j} = 1$. Hence $N(z) > N(z')$.

Iterating this process gives a sequence $z \lneqq z' \lneqq z'' \lneqq \ldots$ in S with $N(z) > N(z') > N(z'') > \ldots$, which therefore must terminate. It can only terminate with a $z^{(m)} \in R$. Hence $w = z^{(m)}$ has all desired properties.

LEMMA 2. R = Convex of T.

(This theorem is due to G. Birkhoff, Rev. Univ. Nac. Tacuman, Series A, Vol. 5 [1946], pp. 147-148. Cf. also G. Birkhoff, "Lattice Theory," Revised Edition, Amer. Math. Soc. Coll. Series, Vol. 25 [1948], example 4* on p. 266. The proof that follows is more direct than Birkhoff's.)

PROOF. R is clearly convex. $R \supseteq T$ is immediate. Hence $R \supseteq$ Convex T.

$R \subseteq$ Convex T is demonstrated, if it is established, that all extreme points of the convex R belong to T. Actually they form precisely the set T.

That every point of T is an extreme point of R is clear: A $z \in T$ belongs to R, and if it were not extreme, then $z = tz' + (1 - t)z''$ with $z', z'' \in R$; $z' \neq z''$; $0 < t < 1$. Choose $z'_{ij} \neq z''_{ij}$, say $z'_{ij} < z''_{ij}$. Then $z_{ij} = tz'_{ij} + (1 - t)z''_{ij}$, hence $z'_{ij} < z_{ij} < z''_{ij}$. Now either $z_{ij} = 0$, implying $z'_{ij} < 0$, or $z_{ij} = 1$, implying $z''_{ij} > 1$ -- and both are impossible.

There remains, therefore, only this: To prove that every extreme point of R belongs to T. This is shown as follows:

For a $z \in R$ call a pair i,j <u>inner</u>, if $z_{ij} \neq 0, 1$. Clearly $z \in T$ (for a $z \in R$) means that all $z_{ij} = 0, 1$, i.e., that no i,j is inner. Hence $z \notin T$ means that inner i,j exist.

If a line i (or a column j) contains at most one inner element z_{ij}, then $z_{ij} = 1 - \sum_{j'} z_{ij'}$ (or $z_{ij} = 1 - \sum_{j'} z_{i'j}$) is necessarily $= 1, 0, -1, -2, \ldots$. Since $z_{ij} \geqq 0$, therefore $z_{ij} = 0, 1$, i.e., i,j is not inner either. In other words: If i,j is inner, then there exists an inner i,j' (i',j) with $j' \neq j$ $(i' \neq i)$.

Now consider a $z \in R$, such that $z \notin T$. Let i,j be inner. Choose $j' \neq j$ such that i,j' is inner, then $i' \neq i$ such that i',j' is inner, then $j'' \neq j'$ such that i',j'' is inner, then $i'' \neq i'$ such that i'',j'' is inner, etc. In this way two sequences $i, i', i'', \ldots$ and $j, j', j'', \ldots$ arise, such that $i^{(m)},j^{(m)}$ is inner, $i^{(m)},j^{(m+1)}$ is inner, and $i^{(m)} \neq i^{(m+1)}$, $j^{(m)} \neq j^{(m+1)}$ (all this for all $m = 0, 1, 2, \ldots$). Hence $i^{(p)} = i^{(q)}$ or $j^{(p)} = j^{(q)}$ must occur sometime for $p \neq q$. Choose such a pair with $p < q$, and with q as small as possible, and with p (for this q) as large as possible. Hence $i^{(p)}, i^{(p+1)}, \ldots, i^{(q-1)}, i^{(q)}$ are pairwise different, also $j^{(p)}, j^{(p+1)}, \ldots, j^{(q-1)}, j^{(q)}$ are pairwise different, with the possible exception of $i^{(p)} = i^{(q)}$ or $j^{(p)} = j^{(q)}$ (or both). For $j^{(p)} = j^{(q)}$ define $i_o = i^{(p)}$, $j_o = j^{(p)}$ $(= j^{(q)})$, $i_1 = i^{(p+1)}$, $j_1 = j^{(p+1)}$, $\ldots$, $i_{q-p-1} = i^{(q-1)}$, $j_{q-p-1} = j^{(q-1)}$. For $j^{(p)} \neq j^{(q)}$, hence necessarily $i^{(p)} = i^{(q)}$, define $i_o = i^{(p)}$ $(= i^{(q)})$, $j_o = j^{(q)}$, $i_1 = i^{(p+1)}$, $j_1 = j^{(p+1)}$, $\ldots$, $i_{q-p-1} = i^{(q-1)}$, $j_{q-p-1} = j^{(q-1)}$. Thus two sequences $i_o, i_1, \ldots, i_{s-1}$ and $j_o, j_1, \ldots, j_{s-1}$ $(s = q - p$, also define $j_s = j_o)$ arise, with the following properties: $i_o, i_1, \ldots, i_{s-1}$ are pairwise different, also $j_o, j_1, \ldots, j_{s-1}$ are pairwise different, i_t,j_t is inner, i_t,j_{t+1} is inner (all this for $t = 0, 1, \ldots, s - 1$). I.e., the quantities

(1) $z_{i_oj_o}, z_{i_1j_1}, \ldots, z_{i_{s-1}j_{s-1}}$,

(2) $z_{i_oj_1}, z_{i_1j_2}, \ldots, z_{i_{s-1}j_s}$ $\quad (j_s = j_o)$

are all $> 0, < 1$.

Now let ε be the minimum of the quantities in the lines (1), (2). Then $\varepsilon > 0$. Define $z' = (z'_{ij})$ and $z'' = (z''_{ij})$ as follows:

$$z'_{ij}(z''_{ij}) \begin{cases} = z_{ij} + \varepsilon\ (z_{ij} - \varepsilon) & \text{for the } i,j \text{ of line (1)} \\ = z_{ij} - \varepsilon\ (z_{ij} + \varepsilon) & \text{for the } i,j \text{ of line (2)} \\ = z_{ij} & \text{otherwise} \end{cases} \Bigg| \; .$$

Then $z' \in R$ and $z'' \in R$ are readily verified. Also clearly $z' \neq z''$ and $z = \frac{1}{2} z' + \frac{1}{2} z''$.

Hence z is not an extreme point in R, q.e.d. ———

We now return to the 2-dimensional game and a characterization of player I's optimal strategies. Let $x = (x_{ij})$ be an optimal strategy and let A be the value of the game (for player II). We define

$$z_{ij} = \frac{\alpha_{ij} x_{ij}}{A}, \quad z = (z_{ij}) .$$

Clearly all $\sum_j \alpha_{ij} x_{ij} \leqq A$ and all $\sum_i \alpha_{ij} x_{ij} \leqq A$, i.e., all $\sum_j z_{ij} \leqq 1$ and all $\sum_i z_{ij} \leqq 1$. Hence $z = (z_{ij})$ belongs to S.

Now Lemma 1 implies the existence of a $w = (w_{ij})$ belonging to R, with $z \leqq w$. Form

$$w_{ij} = \frac{\alpha_{ij} u_{ij}}{A}, \quad u = (u_{ij}) .$$

Then all $\sum_j \alpha_{ij} u_{ij} = A \sum_j w_{ij} = A$, and all $\sum_i \alpha_{ij} u_{ij} = A \sum_i w_{ij} = A$, hence $\operatorname{Max}_{i',j'} \left(\sum_j \alpha_{i'j} u_{i'j}, \sum_i \alpha_{ij'} u_{ij'} \right) = A$. Also $z_{ij} \leqq w_{ij}$, hence $x_{ij} \leqq u_{ij}$. Hence $\sum_{ij} x_{ij} \leqq \sum_{ij} u_{ij}$, $\theta = \sum_{ij} x_{ij} / \sum_{ij} u_{ij} \leqq 1$. Put $v_{ij} = \theta u_{ij}$.

Then $\sum_{ij} v_{ij} = \sum_{ij} x_{ij} = 1$, i.e., the v_{ij} can be used as probabilities, like the x_{ij}. Also $\operatorname{Max}_{i',j'} \left(\sum_j \alpha_{i'j} v_{i'j}, \sum_i \alpha_{ij'} v_{ij'} \right) = \theta A$. If $\theta < 1$, then this contradicts the optimality of $x = (x_{ij})$. Hence $\theta = 1$, i.e., $\sum_{ij} x_{ij} = \sum_{ij} u_{ij}$. Since $x_{ij} \leqq u_{ij}$, this implies $x_{ij} = u_{ij}$, i.e., $z_{ij} = w_{ij}$. Hence $z = w \in R$.

Now Lemma 2 implies that z is the center of gravity of certain points of T. I.e.,

$$z = \sum_\nu t_\nu z^\nu , \quad z^\nu \in T , \quad \sum_\nu t_\nu = 1 , \quad t_\nu \geqq 0 .$$

The $t = 0$ may be omitted, i.e., $t > 0$ may be assumed. Form

$$z^\nu_{ij} = \frac{\alpha_{ij} x^\nu_{ij}}{A}, \quad x^\nu = (x^\nu_{ij}) .$$

Then the optimality of x implies that one of all x^ν. Also $z^\nu_{ij} = \delta_{iP^\nu, j}$ for a suitable permutation P^ν. Hence

$$x^\nu_{ij} = \frac{A}{\alpha_{ij}} \delta_{iP^\nu, j} .$$

In other words: All optimal strategies are centers of gravity of optimal strategies of the special form

(*) $$x_{ij} = \frac{A}{\alpha_{ij}} \delta_{i^P, j} \qquad (P \text{ a permutation})$$

Consider, therefore, the strategies of the form (*). For such a strategy all $\sum_j \alpha_{ij} x_{ij} = A$ and all $\sum_i \alpha_{ij} x_{ij} = A$. Hence

$$\operatorname*{Max}_{i', j'} \left(\sum_j \alpha_{i'j} x_{i'j} , \sum_i \alpha_{ij'} x_{ij'} \right) = A .$$

Hence the optimal ones among these strategies are those that give the minimum A.

Now, since the x_{ij} are probabilities, $\sum_{ij} x_{ij} = 1$, i.e., $\sum_i \frac{A}{\alpha_{i,i^P}} = 1$, i.e., $A = 1/\sum_i \frac{1}{\alpha_{i,i^P}}$. Hence the minimum A corresponds to the maximum $\sum_i \frac{1}{\alpha_{i,i^P}}$. I.e., precisely those permutations P give the optimal strategies in question, for which $\sum_i \frac{1}{\alpha_{i,i^P}}$ assumes its maximum value.

To sum up:

THEOREM. The extreme optimal strategies (i.e., those, of which all others are centers of gravity) of the 2-dimensional game are precisely the following ones: Consider those permutations P which maximize

$$\sum_i \frac{1}{\alpha_{i,i^P}} .$$

For each P of this class form the strategy $x = (x_{ij})$ according to (*) above.———

Note, that this means that player I plays only those cells where the permutation matrix (of P) has a 1. (Here line guesses (i) and column guesses (j) correspond to each other equivalently under the relation $j = i^P$.) His play among these cells is then determined by the 1-dimensional game. ———

The condition expressed in the above Theorem for P is exactly the optimal assignment problem with $a_{ij} = 1/\alpha_{ij}$, where the a_{ij} are the elements of the assignment matrix (which, we saw, could be considered as all positive).

Several further remarks can be made.

1) A transformation of $a_{ij} \longrightarrow a_{ij} + u_i + v_j$ in the assignment matrix leaves the solution unchanged, and hence the game will be invariant (in its P's) under the corresponding

$$\alpha_{ij} \longrightarrow \frac{\alpha_{ij}}{1 + \alpha_{ij}(u_i + v_j)} .$$

That the game should be so invariant is not at all clear initially from the game itself. (Note, this is not complete invariance. The 1-dimensional solutions for a particular P may change, though the P remains the same.)

2) Various extensions of the optimum assignment problem are possible and can be settled in essentially the same manner. Thus one can specify certain many-to-many assignment patterns between persons and jobs, and the like.

In addition, certain formal generalizations of the game are possible -- to various k-dimensional forms with $k = 3, 4, \ldots$. These seem to be interesting, but present serious difficulties.

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