CONTRIBUTIONS TO THE THEORY OF GAMES

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SOLUTIONS OF GAMES BY DIFFERENTIAL EQUATIONS*

G. W. Brown and J. von Neumann

§1.

The purpose of this note is to give a new proof for the existence of a "value" and of "good strategies" for a zero-sum two-person game. This proof seems to have some interest because of two distinguishing traits:

(a) Although the theorem to be proved is of an algebraical nature, a very simple proof is obtained by analytical means.

(b) The proof is "constructive" in a sense that lends itself to utilization when actually computing the solutions of specific games. The procedure could be "mechanized" with relative ease, both for "digital" and for "analogy" methods. In the latter case it is probably much less sensitive to the precision of the equipment, than the somewhat related problem of "linear equation solving" or "matrix inversion."

The derivations which follow are based on results that were obtained independently by the two authors. Further results of one of them (G. W. Brown) are published elsewhere [1][1].

§2.

Consider first the special case of a symmetric game, i.e., where the "game matrix" $A_{ij}(i,\ j = 1,\ \ldots,\ m)$ is antisymmetric: $A_{ij} = -A_{ji}$.

Write for vectors, $x = (x_i)$, $u = (u_i)$, and use the notations

$$(1)\qquad \begin{cases} u_i = \sum_j A_{ij} x_j , \\ \varphi(u_i) = \operatorname{Max}(0,\ u_i) , \\ \phi(x) = \sum_i \varphi(u_i) , \\ \Psi(x) = \sum_i (\varphi(u_i))^2 . \end{cases}$$

Consider the differential equation system

$$(2)\qquad \frac{dx_i}{dt} = \varphi(u_i) - \phi(x) \cdot x_i ,$$

[1] Numbers in square brackets refer to the bibliography at the end of this paper.

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starting with a vector

(3) $$x^0 = (x_i^0),\ x_i^0 \geq 0,\ \sum_i x_i^0 = 1\ .$$[2]

$x_i = 0$ implies $\frac{dx_i}{dt} = \varphi(u_i) \geq 0$, hence $x_i \geq 0$ can never go over into $x_i < 0$, i.e., always

(4) $$x_i \geq 0\ .$$

Summing over all i gives

$$\frac{d}{dt}\left(\sum_i x_i\right) = \phi(x)\left(1 - \sum_i x_i\right)\ ,$$

i.e.,

$$\frac{d}{dt} \ln \left|1 - \sum_i x_i\right| = -\phi(x)\ ,$$

hence $\sum_i x_i = 1$ can never go over into $\sum_i x_i \neq 1$, i.e., always

(5) $$\sum_i x_i = 1\ .$$

Next, when $\varphi(u_i) > 0$, then

$$\frac{d\varphi(u_i)}{dt} = \sum_j A_{ij} \frac{dx_j}{dt} = \sum_j A_{ij}\, \varphi(u_j) - \phi(x) \sum_j A_{ij} x_j\ ,$$

hence always

$$\frac{d(\varphi(u_i))^2}{dt} = 2 \sum_j A_{ij}\, \varphi(u_i)\, \varphi(u_j) - 2\, \phi(x) \sum_j A_{ij}\, \varphi(u_i) x_j\ .$$

Summing over all i gives

$$\frac{d\Psi(x)}{dt} = 2 \sum_{ij} A_{ij}\, \varphi(u_i)\, \varphi(u_j) - 2\, \phi(x) \sum_{ij} A_{ij}\, \varphi(u_i) x_j\ .$$

The first term on the right-hand side vanishes, because A_{ij} is antisymmetric. The second term is equal to

$$\sum_i \varphi(u_i) \sum_j A_{ij} x_j = \sum_i \varphi(u_i) u_i\ .$$

[2]Existence of a unique solution to this system is assured by virtue of the fact that the system is piecewise well-behaved, with matching of first derivatives at the boundaries. The related system obtained by dropping the last term in (2) is piecewise a linear system, and has a growing solution, proportional to the solution of (2). The last term in (2) simply normalizes the solution to make (5) hold.

Whenever $\varphi(u_i) \neq 0$, then $\varphi(u_i) = u_i$. Hence the above expression is equal to $\sum_i (\varphi(u_i))^2 = \Psi(x)$. Therefore

$$\frac{d\Psi(x)}{dt} = -2\,\phi(x)\,\Psi(x)\,. \tag{6}$$

Now clearly

$$(\Psi(x))^{\frac{1}{2}} \leq \phi(x) \leq (m\Psi(x))^{\frac{1}{2}}\,. \tag{7}$$

Hence as long as $\Psi(x) > 0$, also $\phi(x) > 0$, and $\Psi(x)$ is decreasing; also

$$\frac{d\Psi(x)}{dt} \leq -2\,(\Psi(x))^{\frac{3}{2}}, \quad -\frac{1}{2}\,(\Psi(x))^{-\frac{3}{2}}\,\frac{d\Psi(x)}{dt} \geq 1\,,$$

hence

$$(\Psi(x))^{-\frac{1}{2}} \geq (\Psi(x_0))^{-\frac{1}{2}} + t\,,$$

$$\Psi(x) \leq \frac{\Psi(x_0)}{\left(1 + (x_0)^{\frac{1}{2}}\,t\right)^2}\,. \tag{8}$$

If ever $\Psi(x) = 0$, then this remains true from then on (i.e., for all larger t), and so (8) is true again.

Finally from (7), (8)

$$\phi(x) \leq \frac{m^{\frac{1}{2}}\,\Psi(x_0)^{\frac{1}{2}}}{1 + \Psi(x_0)^{\frac{1}{2}}\,t} \tag{9}$$

and from (8)

$$\varphi(u_i) \leq \frac{\Psi(x_0)^{\frac{1}{2}}}{1 + \Psi(x_0)^{\frac{1}{2}}\,t}\,. \tag{10}$$

§ 3.

By (8), (9), (10) $t \longrightarrow +\infty$ implies

$$\Psi(x) \longrightarrow 0,\ \phi(x) \longrightarrow 0\,,$$

and all

$$\varphi(u_i) \longrightarrow 0\,.$$

That the x_i themselves have limits for $t \longrightarrow +\infty$ is not clear; (2) and (10) do not seem to suffice to prove this. Nevertheless, since the range (4), (5) of the x_i is compact, limit points $x^\infty = (x_i^\infty)$ of the $x = (x_i)$ for $t \longrightarrow +\infty$ must exist. For any such x^∞ (4), (5) must again be true, and (10) gives that all

$$\varphi(u_i^\infty) = 0 \ ,$$

i.e., all

$$\sum_j A_{ij} x_j^\infty \leq 0 \ . \tag{11}$$

Hence any $x^\infty = (x_i^\infty)$ represents a "good strategy," and the "value" of the (symmetric) game is, as it should be, zero.

§4.

Consider next an arbitrary game, i.e., one with an unrestricted "game matrix" B_{kl} $(k = 1, \ldots, p;\ l = 1, \ldots, q)$. Various ways of reducing this to a symmetric game are known. They differ from each other, among other things, in the order m of the symmetric game, i.e., of the antisymmetric matrix $A_{ij}(i, j = 1, \ldots, m)$ to which they lead. A very elegant method of this type has been lately found [2] which gives the remarkably low value $m = p + q + 1$. One of the authors (J. von Neumann) had obtained earlier another method, which gives the larger value $m = pq$. (This is referred to, but not described, in [3], page 168.) We will follow here the second procedure, partly because its underlying qualitative idea is simpler, and partly because, although its m is considerably larger than that of the other method (pq vs. $p + q + 1$, cf. above), it leads ultimately to a set of differential equations in fewer variables ($p + q - 2$ vs. $p + q$, cf. the remarks at the end of §5.).

The qualitative idea behind the method referred to is this: Assume that a player knew how to play every conceivable symmetric game A. Assume that he were asked to play a (not necessarily symmetric) game B. How could he then reduce it to known (symmetric) patterns?

He could do it like this: He could imagine that he is playing (simultaneously) two games B: Say B' and B". In B' he has the role of the first player, in B" he has the role of the second player — for his opponent the positions are reversed. The total game A, consisting of B' and of B" together, is clearly symmetric. Hence the player will know how to play A — hence also its parts, say B', i.e., B.

In spite of the apparent "practical" futility of this "reduction," it nevertheless expresses a valid mathematical procedure. The mathematical procedure is as follows:

Let k', l' be the indices k, l for the game B', and k'', l'' those for the game B''. The player under consideration then controls the indices k', l'', and his opponent controls k'', l'. Hence i may be made to correspond to the pair (k', l''), and j to the pair (k'', l'). The game matrices are $B_{k'l'}$ for B' and $-B_{k''l''}$ for B'', i.e. $B_{k'l'} - B_{k''l''}$ for A, i.e.

(12) $A_{ij} = B_{k'l'} - B_{k''l''}$, where $i = (k', l'')$, $j = (k'', l')$.

The symmetry of this new game, i.e. the antisymmetry of A_{ij}, is obvious. Clearly $m = pq$.

Hence a system $x = (x_i) = (x_{kl})$ exists, such that all

$$x_i \geq 0, \sum_i x_i = 1 ,$$

and all

$$\sum_j A_{ij} x_j \leq 0$$

(cf. (4), (5), (11)). This means

$$x_{kl} \geq 0, \sum_{kl} x_{kl} = 1 ,$$

and

$$\sum_{k''l'} (B_{k'l'} - B_{k''l''}) x_{k''l'} \leq 0 ,$$

i.e.

$$\sum_{k''l'} B_{k'l'} x_{k''l'} \leq \sum_{k''l'} B_{k''l''} x_{k''l'} ,$$

$$\sum_{l'} B_{k'l'} (\sum_{k''} x_{k''l'}) \leq \sum_{k''} B_{k''l''} (\sum_{l'} x_{k''l'}) .$$

Putting

(13) $$\xi_k = \sum_l x_{kl}, \eta_l = \sum_k x_{kl} ,$$

these inequalities yield

(14) $$\xi_k \geq 0, \quad \eta_l \geq 0 ,$$

(15) $$\sum_k \xi_k = 1, \quad \sum_l \eta_l = 1 ,$$

and

$$\sum_{l'} B_{k'l'} \eta_{l'} \leq \sum_{k''} B_{k''l''} \xi_{k''} ,$$

i.e.

$$\text{Max}_{k} \sum_{l} B_{kl}\eta_l \leq \text{Min}_{l} \sum_{k} B_{kl}\xi_k . \tag{16}$$

(14), (15), (16) imply, of course, that equality holds in (16), that $\xi = (\xi_k)$ and $\eta = (\eta_l)$ are "good strategies" for the two players, and that the common value of both sides of (16) is the "value" of the (original, not necessarily symmetric) game. (Cf. [3], pp. 153 and 158.)

§ 5.

Apply now the differential equation system (2) to the "derived" game (12). Restating (1), (2) gives

$$u_{k'l''} = \sum_{k''l'} (B_{k'l'} - B_{k''l''})\, x_{k''l'} ,$$

$$\varphi(u) = \text{Max}\,(0, u)$$

$$\phi(x) = \sum_{kl} \varphi(u_{kl})$$

$$\Psi(x) = \sum_{kl} (\varphi(u_{kl}))^2$$

and

$$\frac{dx_{kl}}{dt} = \varphi(u_{kl}) - \phi(x)\, x_{kl} .$$

This system involves $m = pq$ variables x_{kl}. It can, however, be "contracted" as follows:

Clearly

$$u_{k'l''} = \sum_{l'} B_{k'l'}\eta_{l'} - \sum_{k''} B_{k''l''}\xi_{k''} ,$$

i.e.

$$u_{kl} = v_k - w_l ,$$

where

$$\begin{cases} v_k = \sum_{l} B_{kl}\eta_l , \\ w_l = \sum_{k} B_{kl}\xi_k . \end{cases} \tag{17}$$

$\varphi(u)$ may be defined as before:

$$\varphi(u) = \text{Max}\,(0, u) . \tag{18}$$

$\phi(x), \Psi(x)$ depend no longer on all pq components of $x = (x_i) = (x_{kl})$, but only on the $p + q$ components of $\xi = (\xi_k)$ and $\eta = (\eta_l)$:

$$(19)\quad \begin{cases} \phi(\xi, \eta) = \sum_{kl} \varphi(v_k - w_l) , \\ \psi(\xi, \eta) = \sum_{kl} (\varphi(v_k - w_l))^2 . \end{cases}$$

Summing the x_{kl}-differential-equations over all l gives

$$(20)\quad \frac{d\xi_k}{dt} = \sum_l \varphi(v_k - w_l) - \phi(\xi, \eta)\,\xi_k ,$$

summing them over all k gives

$$(21)\quad \frac{d\eta_l}{dt} = \sum_k (v_k - w_l) - \phi(\xi, \eta)\,\eta_l .$$

Thus a system (20), (21) has been obtained, which involves only $p + q$ variables ξ_k and η_l.

Combining the observations of §3. and those at the end of §4. shows, since the $\xi = (\xi_k)$, $\eta = (\eta_l)$ vary in the compact joint range defined by (14), (15), that they possess (joint) limiting points $\xi^\infty = (\xi_k^\infty)$, $\eta^\infty = (\eta_l^\infty)$. Any such pair represents a pair of "good strategies."

Because of (15), the number of variables involved in (2) is not m, but $m - 1$. Because of (15), the number of variables involved in (20), (21) is not $p + q$, but $p + q - 2$.

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