Jenö Egerváry: from the origins of the Hungarian algorithm to satellite communication

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Abstract We discuss some relevant results obtained by Egerváry in the early Thirties, whose importance has been recognized several years later. We start with a quite well-known historical fact: the first modern polynomial-time algorithm for the assignment problem, invented by Harold W. Kuhn half a century ago, was christened the "Hungarian method" to highlight that it derives from two older results, by Kőnig (1916) and Egerváry (1931). (A recently discovered posthumous paper by Jacobi (1804-1851) contains however a solution method that appears to be equivalent to the Hungarian algorithm.) Our second topic concerns a combinatorial optimization problem, independently defined in satellite communication and in scheduling theory, for which the same polynomial-time algorithm was independently published thirty years ago by various authors. It can be shown that such algorithm directly implements another result contained in the same 1931 paper by Egerváry. We finally observe that the latter result also implies the famous Birkhoff-von Neumann theorem on doubly stochastic matrices.

Key words Assignment problem, Hungarian algorithm, Open shop scheduling, Satellite switched time division multiple access

1 Introduction

This paper is devoted to a discussion on the implications of a paper that was published in Hungarian by Jenő Egerváry [6] in 1931, and was ignored for many years by the international mathematical community. The paper, translated into English by Harold W. Kuhn [20] in 1955, basically consists of two theorems. The first one is a cornerstone of the famous "Hungarian method" for the assignment problem. The proof of the second (much less known) theorem provides a polynomial-time algorithm for a combinatorial optimization problem that was independently established in the Seventies in two different domains, for which different authors proposed solution algorithms that replicate the one defined by Egerváry. In addition, an easy extension of the second theorem to non-negative real matrices contains as a special case the celebrated Birkhoff–von Neumann theorem, that was published in Spanish by Garrett Birkhoff [1] in 1946 and for which John von Neumann [37] gave in 1953 an elegant elementary proof.

In Section 2 we recall the early years of the assignment problem and the contributions of the Hungarian mathematicians. More details on these exciting events can be found in Kuhn [22], Schrijver [32] (Chapter 17), Frank [8], Jüttner [18], Schrijver [33] and Burkard, Dell'Amico and Martello [2] (Chapters 2–4). We conclude the section by briefly reviewing the recent historical discovery of a posthumous paper by Jacobi that anticipates by many decades the Hungarian algorithm.

In Section 3 we consider two equivalent problems, arising in satellite communication and in scheduling theory, and discuss two well-known algorithms independently proposed by different authors. It turns out that such algorithms are equivalent, and implement a technique developed by Egerváry for proving a second theorem given in the same 1931 paper. We conclude with the observation that the latter Egerváry's result also implies the famous Birkhoff-von Neumann theorem on doubly stochastic matrices. More details on these topics can be found in Dell'Amico and Martello [3].

2 Assignment problem

The *Assignment Problem* (AP) has a very simple definition: Given an $n \times n$ matrix $A = (a_{ij})$, find a permutation φ of the integers $1, 2, \ldots, n$ that maximizes (or, alternatively, minimizes)

$$\sum_{i=1}^{n} a_{i\varphi(i)}.$$

The problem is equivalently defined by the classical mathematical model

$$\max \quad \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_{ij} \tag{1}$$

$$\text{s.t.} \quad \sum_{j=1}^{n} x_{ij} = 1 \qquad (i = 1, 2, \ldots, n), \tag{2}$$

$$\sum_{i=1}^{n} x_{ij} = 1 \qquad (j = 1, 2, \ldots, n), \tag{3}$$

$$x_{ij} \in \{0, 1\} \qquad (i, j = 1, 2, \ldots, n). \tag{4}$$

In spite of its simplicity, in the last fifty years this problem attracted hundreds of researchers, accompanying and sometimes anticipating the development of Combinatorial Optimization.

The origins of the assignment problem date back to the Eighteenth century, when Monge [28] formulated a continuous mass transportation problem as a huge AP that minimizes the cost for transporting all the molecules. The combinatorial structure of the problem was investigated in the early Twentieth century (Miller [27], Kőnig [25], Frobenius [10], Egerváry [6]), as we will see in the present section, while the first (exponential-time) implicit enumeration algorithm was proposed in the Forties (Easterfield[4]). However, the problem was formulated in a modern way in the Fifties, not by a mathematician, but by psychologist. In 1950 Robert L. Thorndike [36], President of the Division on Evaluation and Measurement of the American Psychological Association, defined the AP in the same way teachers use nowadays to describe it to students:

> *Given a set of ... N job vacancies to be filled, and N individuals to be used in filling them, assign the individuals to the jobs in such a way that the average success of all the individuals in all the jobs to which they are assigned will be a maximum*

The problem name, though, was not invented by Thorndike, but by Votaw and Orden [38] in a 1952 paper titled "The personnel assignment problem".

Thorndike reports that he presented the problem to a mathematician, who observed that there is a finite number of permutations in the assignment of men to jobs, so from the point of view of the mathematician there was no problem: one had only to try them all and choose the best. He dismissed the problem at that point, and Thorndike observes:

> *This is rather cold comfort to the psychologist, however, when one considers that only ten men and ten jobs mean over three and a half million permutations. Trying out all the permutations may be a mathematical solution to the problem, it is not a practical solution.*

It is funny that a psychologist of the Fifties could apprehend a complexity issue better than a mathematician. (Mathematicians will indeed understand complexity 15 years later with Jack Edmonds [5].) Also note that a modern personal computer could solve the "ten men and ten jobs" instance by complete enumeration, but no supercomputer on Earth could enumerate a "twenty-five men and twenty-five jobs" instance within any acceptable time.

In his reminiscences on the origin of the Hungarian method, Kuhn [22] reminds that he was attracted to the problem in 1953, when C.B. Tompkins (1912–1971), a pioneer in numerical analysis and computing, was unsuccessfully trying to program a SWAC (Standards Western Automatic Computer) to solve small-size AP instances. In that period Kuhn was reading the classic graph theory textbook by Dénes Kőnig [26], *Theorie der Endlichen und Unendlichen Graphen*, and encountered his augmenting path algorithm for the matching problem on bipartite graphs. We need here to go further back to the past, and examine some fundamental results obtained in the early years of the Twentieth Century.

2.1 Kőnig's Theorem

Let $G = (U, V; E)$ be a bipartite graph. A *matching* is a subset M of E such that every vertex of G coincides with at most one edge of M. If $|U| = |V| = n$ a matching of cardinality n is called *perfect.* A *vertex cover* is a subset C of $U \cup V$ such that every edge of G coincides with at least one vertex of C.

In 1916 Kőnig [25] gave a constructive proof of the following

Theorem 1 (Kőnig [25], 1916) *In a bipartite graph the maximum cardinality of a matching is equal to the minimum cardinality of a vertex cover.*

The problem of finding the matching of maximum cardinality can be modeled as a linear program whose constraint matrix is the adjacency matrix of the bipartite graph. Hence this beautiful theorem can be seen as the first duality result in linear programming. The celebrated lemma proved in 1902 by Farkas [7] (another Hungarian mathematician) was a first step in this direction, but limited to a feasibility property. The Egerváry's Theorem, discussed in Section 2.2, will then extend the Kőnig's result to the weighted case, thus fully developing the dual aspect of the problem.

A result equivalent to Theorem 1 (formulated as a property on the decomposition of matrices) had been proven in 1912 by Frobenius [9] through algebraic considerations. However, the most notable aspect of Kőnig's contribution is his proving technique. It is clear that $|M| \leq |C|$ holds for any matching M and vertex cover C (for any vertex of C, at most one emanating edge can belong to a matching). In order to show that equality holds, i.e., that there is a matching M whose cardinality is equal to that of a minimum vertex cover, Kőnig gave a constructive proof that can be summarized as:

1. given any non maximum (possibly empty) matching, there is an algorithm to produce a new matching with cardinality increased by 1;
2. if the algorithm fails then there exists a vertex cover having the same cardinality as the current matching.

Although described in a slightly different way (by associating a directed graph to G), the algorithm of point 1. is nothing else than the well-known *augmenting path algorithm.* Given a matching M in G, the algorithm finds a path formed by an odd number of edges of E, such that all edges in odd position do not belong to M and all edges in even position belong to M. By interchanging matching and non-matching edges along the path one obtains a new matching with one more edge.

It can be shown (see, e.g., Burkard, Dell'Amico and Martello [2]) that Theorem 1 implies the famous *marriage theorem*, proved in 1935 by Hall [14]. For any vertex $i \in U$, let $N(i)$ denote the set of its neighbors: $N(i) = \{j \in V \, : \, [i, j] \in E\}$. For any subset $U' \subseteq U$, let $N(U') = \bigcup_{i \in U'} N(i)$. Then

Theorem 2 (Hall [14], 1935) *A bipartite graph $G = (U, V; E)$ with $|U| = |V|$ admits a perfect matching if and only if*

$$|U'| \leq |N(U')| \tag{5}$$

holds for all subsets U' of U.

2.2 Egerváry's Theorem I

Let us return to the Fifties, and to Kuhn reading Kőnig's book. The matching problem is the special case of assignment problem that arises when the cost matrix is binary. A footnote in Kőnig's book mentioned a 1931 paper by Egerváry [6] published, in Hungarian, in *Matematikai és Fizikai Lapok* (Mathematical and Physical Pages). According to Kőnig, this paper extended his result to the weighted case. Kuhn then borrowed from his library a Hungarian dictionary and a grammar, and wrote an English translation of the paper, that was published as a technical report of the George Washington University (Kuhn [20]).

Egerváry's analysis is based on covering systems. Given an $n \times n$ matrix $A = (a_{ij})$, a *covering system* is a set of *lines* (rows and columns) that contain the ith row with multiplicity λ_i and the jth column with multiplicity μ_j, and satisfy

$$\lambda_i + \mu_j \geq a_{ij} \quad (i, j = 1, 2, \ldots, n). \tag{6}$$

A covering system of minimum value

$$\sum_{k=1}^{n} (\lambda_k + \mu_k) \tag{7}$$

is called a *minimal covering system*. It immediately appears that the minimization of (7) subject to (6) is, in modern terminology, the dual problem associated with the primal model (1)-(4), with dual variables (λ_i) and (μ_j) associated with constraints (2) and (3), respectively.

The main result proved by Egerváry is indeed the following:

Theorem 3 (Egerváry [6], 1931) *If (a_{ij}) is an $n \times n$ matrix of non-negative integers then, subject to condition (6), we have*

$$\min \sum_{k=1}^{n} (\lambda_k + \mu_k) = \max_{\varphi} \sum_{i=1}^{n} a_{i\varphi(i)}.$$

In other words, the primal and the dual problem have the same solution value.

Given a complete bipartite graph $G = (U, V; E)$ with $|U| = |V| = n$, let a_{ij} be the weight of edge $[i, j] \in E$ $(i, j = 1, 2, \ldots, n)$. Define the weight of a matching M in G as $w(M) = \sum_{[i,j] \in M} a_{ij}$. It is clear that $\sum_{k=1}^{n} (\lambda_k + \mu_k) \geq$

$w(M)$ holds for any perfect matching M and any pair $((\lambda_i), (\mu_j))$ satisfying (6). Hence

$$\min \sum_{k=1}^{n} (\lambda_k + \mu_k) \geq \max_{\varphi} \sum_{i=1}^{n} a_{i\varphi(i)}. \tag{8}$$

Differently from Kőnig, Egerváry did not provide an explicit algorithm to prove that equality holds in (8). However, his proof implicitly defines the following iterative method:

1. **initialize** with any (λ_i) and (μ_j) satisfying $\lambda_i+\mu_j \geq a_{ij} (i, j = 1, 2, \ldots, n)$;
2. find a maximum matching M (**Theorem** 1) in the subgraph $G(\lambda, \mu)$ of G that only contains the edges of E that satisfy $\lambda_i + \mu_j = a_{ij}$;
3. **if** M is perfect **then** its weight is $w(M) = \sum_{k=1}^{n} (\lambda_k + \mu_k)$: **stop**;
4. **else** $G(\lambda, \mu)$ must contain (**Theorem** 2) a subset $|U'| \subseteq U$ such that $|U'| > |N(U')|$: update the covering system through (**Theorem** 3)

$$\begin{cases} \lambda_i := \lambda_i - 1 \text{ for } i \in U'; \\ \mu_j := \mu_j + 1 \text{ for } j \in N(U'), \end{cases} \tag{9}$$

   thus decreasing the value of $\sum_{k=1}^{n} (\lambda_k + \mu_k)$ by $|U'| - |N(U')| > 0$, and **go to 2**.

From an algorithmic point of view it is easy to observe that the two 1s in (9) can be replaced by $\min\{\lambda_i + \mu_j - a_{ij} : i \in U', j \in N(U')\}$ for a more effective update of the covering system. This is the basic structure of the algorithm developed by Kuhn [19,21]. Kuhn christened it the *Hungarian method*, in honor of these two great Hungarian mathematicians, who also shared a common tragic fate. In 1944, in the period of the anti-Semitic atrocities that took place after the Nazi occupation, fearing imprisonment for his being Jewish, Dénes Kőnig committed suicide. In 1958, in the period of the Communist repression that followed the 1956 revolution, fearing imprisonment for specious accusations, Jenő Egerváry committed suicide. (See Gallai [12], Rózsa [31], Galántai [11] and Rapcsák [30] for further biographical information.)

2.3 Is the Hungarian algorithm a German algorithm?

We briefly mention in this section a recent historical discovery that connects the Hungarian algorithm to a posthumous paper by Jacobi [16], written in Latin and titled *De investigando ordine systematis aequationum differentialum vulgarium cujuscunque*. (On the research of the order of a system of arbitrary ordinary differential equations. An English translation has been provided by Ollivier, see [17].) Jacobi introduces a bound on the order of a system of m ordinary differential equations in m unknowns, and observes that its computation can be reduced to the following problem, that is interesting to read in its original formulation. Preliminary observe that Jacobi, who died in 1851, did not even have a proper terminology. Although matrices and Latin squares were already known by Chinese mathematicians more than two thousands years ago, the term *matrix* was coined in 1850 by Sylvester [35] in a sentence that nicely explains his choice of this word:

> *... an oblong arrangement of terms consisting, suppose, of m lines and n columns. This will not in itself represent a determinant, but is, as it were, a Matrix out of which we may form various systems of determinants by fixing upon a number p, and selecting at will p lines and p columns, the squares corresponding to which may be termed determinants of the p-th order.*

Jacobi gave thus the following definition, in which however the assignment problem is easily recognized:

> *Disponantur nn quantitates $h_k^{(i)}$ quaecunque in schema Quadrati, ita ut habeantur n series horizontales et n series verticales, quarum quaeque est n terminorum. Ex illis quantitatibus eligantur n transversales, i.e. in seriebus horizontalibus simul atque verticalibus diversis positae, quod fieri potest $1 \cdot 2 \cdot \ldots \cdot n$ modis; ex omnibus illis modis quaerendum est is, qui summam n numerorum electorum suppeditet maximam.*

(Arrange nn arbitrary numbers $h_k^{(i)}$ in a square table so that there are n horizontal series and n vertical series, each having n numbers. Among these numbers, we want to select n *transversals*, i.e., all placed in different horizontal and vertical series, which may be done in $1 \cdot 2 \cdot \ldots \cdot n$ ways; among all these ways, we want to find one that gives the maximum sum of the n selected numbers.)

Not only Jacobi defined the AP, but, more importantly, he gave a polynomial-time algorithm to solve it. Although a thorough analysis of this method has not been provided yet, Ollivier and Sadik [29] recently observed that it is essentially identical to the Hungarian algorithm, thus anticipating by many decades the results we have examined in the previous sections. As wittily observed by Kuhn [24], this makes another application of "Stigler's law of eponymy": *No scientific discovery is named after its original discoverer* (Stigler [34], see also Kuhn [23]).

3 Open shop and satellite communication

In this section we show that the proof of the second theorem in the Egerváry [6] paper provides a polynomial-time algorithm that anticipates by several decades algorithms re-discovered in the Seventies in two different research areas.

3.1 Open shop scheduling and time slot assignment

The following problem arises in scheduling theory. We are given m machines and n jobs. Each job requires processing on every machine (in any order), each machine can process at most one job at a time, and no job can be processed simultaneously on two machines. We will assume, without loss of generality, that $m = n$. A non-negative integer matrix T gives the total amount of time, t_{ij}, job j must be processed on machine i ($i, j = 1, 2, \ldots, n$). Each processing can be interrupted at any time and resumed later. The *Preemptive Open Shop Scheduling Problem* is then to find a feasible schedule such that the completion time of the latest job (*makespan*) is as small as possible. (If preemption is not allowed, i.e., no processing can be interrupted, the problem is strongly $\mathcal{NP}$-hard.)

In satellite based telecommunication systems, a satellite is used to transmit information between n different earth stations. Onboard the satellite there are n transponders, and the interconnections between sending and receiving stations are obtained through an $n \times n$ switch. A specific set of n interconnections is called a *switch mode*, and can be represented by a *permutation matrix* $P = (p_{ij})$, i.e., a binary $n \times n$ matrix with exactly one 1-entry in every row and column. An $n \times n$ non-negative *traffic matrix* T specifies, for each pair (i, j), the total amount of time t_{ij} needed to transmit (through one or more switch modes) the required information from station i to station j. Consider a value t such that $t_{ij} \geq t$ if $p_{ij} = 1$: After applying switch mode P for a time interval of length t, the residual traffic matrix is $T - tP$. This transmission technique is called *Satellite-Switched Time-Division Multiple Access* (SS/TDMA). The *Time Slot Assignment Problem* is to find a sequence of switch modes and the corresponding transmission times such that all the information is transmitted in minimum total time.

It is immediate that the above two descriptions define the same problem. A polynomial-time algorithm for the scheduling version was proposed in 1976 by Gonzalez and Sahni [13]. Independently, Inukai [15] proposed in 1979 a polynomial-time algorithm for the telecommunication version. These algorithms work as follows. (We will use the scheduling terminology.)

Consider the maximum row sum of T (maximum total processing time required by any job),

$$r = \max_{1 \leq i \leq n} \sum_{j=1}^{n} t_{ij},$$

and the maximum column sum (maximum total processing time needed on any machine),

$$c = \max_{1 \leq j \leq n} \sum_{i=1}^{n} t_{ij}.$$

It is clear that

$$t^* = \max(r, c)$$

is a lower bound on the optimal solution value.

It is then possible to define, without affecting the solution value, a modified T matrix in which every line has the same sum t^*. This can be easily obtained by defining, for each row i (resp. column j), the slack $a_i = t^* - \sum_{j=1}^{n} t_{ij}$ (resp. $b_j = t^* - \sum_{i=1}^{n} t_{ij}$), and obtaining a correction matrix S through the following iteration:

for $i := 1$ **to** n **do**
  for $j := 1$ **to** n **do**
    $s_{ij} := \min(a_i, b_j)$;
    $a_i := a_i - s_{ij}$;
    $b_j := b_j - s_{ij}$
  endfor
endfor

The Inukai [15] algorithm can then be briefly described as follows:

1. define the modified traffic matrix $T := T + S$;
2. let $G = (U, V; E)$ be a bipartite graph with $|U| = |V|$ and $E = \{[i, j] : t_{ij} > 0\}$;
3. find a perfect matching in G, and the corresponding permutation matrix P;
4. $\tau := \min\{t_{ij} : p_{ij} = 1\}$, $T := T - \tau P$;
5. **if** T is a zero matrix **then stop else go to 2**.

The Gonzalez and Sahni [13] algorithm starts with the weighted bipartite graph $\widetilde{G} = (U, V; \widetilde{E})$ in which t_{ij} is the weight of edge $[i, j] \in E$ $(i, j = 1, 2, \ldots, n)$, and enlarges it by adding:

(i) n vertices to U (fictitious machines);

(ii) n vertices to V (fictitious jobs), and

(iii) different sets of edges with weights such that, for the resulting graph, the sum of weights of the edges incident with each vertex is t^*.

Steps 2.–5. above are then executed on the enlarged graph. It is not difficult to prove that the two algorithms are perfectly equivalent (see Dell'Amico and Martello [3]).

3.2 Egerváry's Theorem II

Let us now consider the second theorem presented by Egerváry [6]. Given a non-negative integer $n \times n$ matrix A, consider the $n!$ distinct permutation matrices $P^k = (p_{ij}^k)$ (that Egerváry calls *diagonal lines*). A system of permutation matrices which contains the kth matrix, P^k, with multiplicity ν_k

is called a *diagonal covering system* for A if

$$\sum_{k=1}^{n!} \nu_k p_{ij}^k \geq a_{ij} \quad (i, j = 1, 2, \ldots, n). \tag{10}$$

A diagonal covering system of minimum value

$$\sum_{k=1}^{n!} \nu_k$$

is called a *minimal diagonal covering system.* Then

Theorem 4 (Egerváry [6], 1931) *If* (a_{ij}) *is an* $n \times n$ *matrix of non-negative integers then, subject to condition (10), we have*

$$\min \sum_{k=1}^{n!} \nu_k = \max \left(\max_i \sum_{j=1}^{n} a_{ij}, \max_j \sum_{i=1}^{n} a_{ij} \right) (= \alpha),$$

In other words, the minimum number of permutation matrices needed to cover a matrix T (see Section 3.1) is equal to the maximum sum of the elements in a line of T. The proof of Theorem 4 provided by Egerváry implicitly defines an algorithm that operates in two steps:

1. define a *majorant* of A, i.e., a matrix A^* such that

   $$a_{ij}^* \geq a_{ij} \quad (i, j = 1, 2, \ldots, n) \tag{11}$$

   and

   $$\sum_{i=1}^{n} a_{ij}^* = \sum_{j=1}^{n} a_{ij}^* = \alpha \quad (i, j = 1, 2, \ldots, n). \tag{12}$$

2. iteratively subtract from A^* permutation matrices P such that $a_{ij}^* > 0$ if $p_{ij} = 1$ until A^* becomes a zero matrix.

Egerváry's implementation of Step 1 is equivalent to the iteration given in Section 3.1 to define the modified traffic matrix $T+S$ which has all line sums equal. Now consider the permutation matrix P selected at any iteration of Step 2, and let $\delta = \min_{ij}\{a_{ij}^* : p_{ij} = 1\}$. Since the next $\delta - 1$ iterations could select the same matrix, Step 2 can be improved by subtracting from A^*, at each iteration, matrix δP. The resulting algorithm is the Inukai [15] algorithm described in Section 3.1.

3.3 Doubly stochastic matrices

An $n \times n$ non-negative real matrix A satisfying

$$\sum_{i=1}^{n} a_{ij} = \sum_{j=1}^{n} a_{ij} = 1 \quad (i, j = 1, 2, \ldots, n). \tag{13}$$

is called *doubly stochastic*. In 1946 Garret Birkhoff, son of the famous American mathematician George David Birkhoff, published on an Argentinian journal an article [1] in Spanish titled "Tres observaciones sobre el algebra lineal" (Three observations on linear algebra). The most notable of these observations reads as follows:

> *Si una matriz $n \times n$ A satisface (13), entonces es una media aritmética de permutaciones.*

(If an $n \times n$ matrix A satisfies (13) then it is a convex combination of permutation matrices.) In 1953 John von Neumann [37] gave an elegant elementary proof of this theorem, which is generally known as the Birkhoff–von Neumann theorem. This result too, however, had been anticipated by Egerváry.

We have seen that Step 2. of the implicit Egerváry's algorithm of Section 3.2 decomposes an integer matrix A^* satisfying (12) into a sum of permutation matrices. Hence, in its original formulation, it does not apply to a real matrix. However, the simple modification we have discussed (subtract, at each iteration, the current permutation matrix P multiplied by the minimum a^*_{ij} value such that $p_{ij} = 1$) produces an algorithm that decomposes a doubly stochastic matrix into a convex combination of permutation matrices.

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