Linear Programming and Extensions

George B. Dantzig

August 1963

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A REPORT PREPARED FOR

UNITED STATES AIR FORCE PROJECT RAND

The RAND Corporation
1700 MAIN ST. • SANTA MONICA • CALIFORNIA

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PREFACE

The final test of a theory is its capacity to solve the problems which originated it.

This book is concerned with the theory and solution of linear inequality systems. On the surface, this field should be just as interesting to mathematicians as its special case, linear equation systems. Curiously enough, until 1947 linear inequality theory generated only a handful of isolated papers, while linear equations and the related subjects of linear algebra and approximation theory had developed a vast literature. Perhaps this disproportionate interest in linear equation theory was motivated more than mathematicians care to admit by its use as an important tool in theories concerned with the understanding of the physical universe.

Since 1947, however, there have appeared thousands of papers concerned with problems of deciding between alternative courses of action. There can be little doubt that it was the concurrent advances in electronic computers which have made it attractive to use mathematical models in decision-making. Therefore it is not surprising that this field has become, like physics before it, an important source for mathematical problems.

When a decision problem requires the minimization of a linear form subject to linear inequality constraints, it is called a linear program. By natural extension, its study provides further insight into the problem of minimizing a convex function whose variables must satisfy a system of convex inequality constraints. It may be used to study topological and combinatorial problems which may be couched in the form of a system of linear inequalities in discrete-valued variables. It provides a framework for extending many problems of mathematical statistics. This, in brief, is the mathematical scope of the book.

To provide motivation, the first three chapters have been devoted to concepts, origins, and formulation of linear programs. To provide insight into application in a "real" environment, two chapters on application conclude the book.

The viewpoint of this work is constructive. It reflects the beginning of a theory sufficiently powerful to cope with some of the challenging decision problems upon which it was founded.

Many individuals have contributed, each in an important way, to the preparation of this volume. John D. Williams of The RAND Corporation, in his former capacity as head of the Mathematics Department and in his present position as member of the Research Council, has been a constant source of encouragement. At his suggestion, the writing of this book was

initiated as an answer to the many requests that flowed into RAND for information on linear programming.

Much of the theoretical foundation of the field of linear programming has been developed by Professor A. W. Tucker and his associates at Princeton University. Professor Tucker, who took a personal interest in the book, was instrumental in having the manuscript critically reviewed by a committee consisting of leading contributors to the field. Dr. Alan Hoffman of IBM Research reviewed Chapter 10, which deals with a perturbation method to avoid degenerate solutions; here the reader will find Hoffman's famous example that demonstrates the possibility of circling in the simplex algorithm. Professor W. Baumol of the Princeton Economics Department was asked to read Chapter 12 on prices, since he has written many papers and books using linear programming as a tool for the solution of economic problems. Professor Harold Kuhn of the Princeton Mathematics and Economics Departments reviewed Chapters 14, 15, and 16, which deal with the transportation problem. Throughout the book there are frequent references to Professor Kuhn's fundamental contributions to the field. Dr. Ralph Gomory of IBM Research attended to Chapter 26, in which his recent, exciting theory of integer programming is presented. The final member of the review committee was Dr. Michel Balinski, a member of the staff of *Mathematica*. Dr. Balinski has a fine grasp of the entire field and worked closely with Professor Tucker on a careful, general review of the volume.

The present content of Chapters 14–21 on transportation and network theory reflects the suggestions of Dr. D. R. Fulkerson of RAND, who kindly reviewed each of the drafts. This particular area has been undergoing rapid development, with Fulkerson a ranking contributor to its elegant theory. I am also pleased to acknowledge indebtedness to Julien Borden, graduate student in mathematics, for his aid in rewriting these chapters.

Individuals who combine a high theoretical ability with a desire to exploit the capabilities of electronic computers contribute in a basic way to the development of the programming field. Such a person is Dr. Philip Wolfe of RAND, who has made fundamental contributions to quadratic, nonlinear, and generalized programming. I am indebted to him for his many constructive suggestions and for his undertaking to rewrite the very important Chapter 3 on formulation, which serves as the key motivation chapter.

Dr. Tibor Fabian, an economist by training, formerly Chief of the Lybrand, Ross Brothers, and Montgomery operations-research team, assisted in the development of the first two chapters on concepts and origins. Professor Paul Randolph of Purdue University played an important role in the development of the earlier drafts of Chapter 5 on the simplex method and of the material on vectors and matrices. At the suggestion of Professor R. Dorfman of Harvard University, Clopper Almon, graduate student in economics at Harvard, undertook to read Chapter 12 on prices and Chapter 23 on the decomposition principle; he kindly contributed § 23-3 and part of § 12-1 illustrating the application of pricing concepts in planning. Similarly William Blattner of U.S. Steel, as part of his graduate studies at the University of California, Berkeley, contributed § 12-4 on sensitivity analysis.

I am grateful to my colleagues at RAND, Dr. Melvin Dresher and Dr. Lloyd Shapley, both experts on game theory, for their suggestions regarding Chapter 13; Dr. Albert Madansky for his many contributions to Chapter 25; and Frank H. Trinkl for his assistance in the organization of Chapter 12.

Marvin Shapiro, formerly of RAND's Computer Sciences Department, and my students, particularly R. Van Slyke, J. Clark, and H. Einstein, carefully read the manuscript and furnished detailed constructive comments. I am grateful to Miss Leola Cutler of RAND for her critical reading of Chapter 18 on bounded variables. The numerical calculations in Chapter 28 were made on RAND's electronic computer, the "Johnniac," by means of a linear programming code developed by W. Orchard-Hays and Miss Cutler.

The administration of the final preparation of the book was done by my very capable assistant, Mrs. Margaret Ryan, who formulated the layout, pre-edited, developed references, and prepared the index. Because of the technical character of the material and the size of the volume, these tasks involved great responsibility. Without her help, the book in its present form would not have been realized.

I am most grateful to Miss Ruth Burns, Chief Secretary of the RAND Mathematics Department, and to her able staff for their full support during the preparation of the manuscript, and to Mrs. Elaine Barth and Mrs. Ella Nachtigal for their work on earlier drafts. It is with great pleasure that I express my gratitude to my secretary, Mrs. Marjorie Romine Marckx, who did much of the final typing and with patience endured my numerous changes in the text.

The editing of the galley and the final page proof was under the jurisdiction of Miss Dorothy Stewart, her assistants at RAND, and my graduate students at the Operations Research Center, University of California, Berkeley: Richard Van Slyke, Donald Steinberg, Earl Bell, Roger Wets, and Mostafa El-Agizy, with Richard Cottle in charge. The detailed index was prepared by Bernard Sussman with the aid of Mrs. Barbara Wade, secretary of the O.R. Center. This team of people uncovered many technical flaws and have contributed in a positive manner to the final polish of the book.

Dr. T. E. Harris, Head, Dr. E. S. Quade, Deputy Head, and Professor E. F. Beckenbach, Editor, of the RAND Mathematics Department kindly provided me with full administrative and editorial support. Likewise, Brownlee W. Haydon, Assistant to the President for Communications at RAND, and John C. Hogan, in charge of RAND publication contracts, gave their full cooperation.

Finally, I am especially grateful to my wife, Anne S. Dantzig, for patience beyond the call of duty. She not only cheerfully suffered my continuous involvement, but even participated actively in various phases of the writing. Many of the better passages of the book reflect her acute rhetorical sense.

GEORGE B. DANTZIG

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CHAPTER 1

THE LINEAR PROGRAMMING CONCEPT

1-1. INTRODUCTION

In the summer of 1949 at the University of Chicago, a conference was held under the sponsorship of the Cowles Commission for Research in Economics; mathematicians, economists, and statisticians from academic institutions and various government agencies presented research using the linear programming tool. The problems considered ranged from planning crop rotation to planning large-scale military actions, from the routing of ships between harbors to the assessment of the flow of commodities between industries of the economy. What was most surprising was that the research reported had taken place during the preceding two years. See Bibliography, [Koopmans, 1951-1].

During and immediately after World War II, work on these and similar problems had proceeded independently until, in 1947, linear programming unified the seemingly diverse subjects by providing a mathematical framework and a computational method, the simplex algorithm, for formulating such problems explicitly and determining their solutions efficiently. This development coincided with the building of electronic digital computers, which quickly became necessary tools in the application of linear programming to areas where hand computation would not have been feasible.

Our immediate purpose is to define mathematical programming in general and linear programming in particular, citing a few typical problems and the characteristics that make them susceptible to solution through the use of linear programming models. Later in the chapter we shall discuss the relation of linear programming to mathematical programming and the relation of mathematical programming to the age of automation that we are approaching.

1-2. THE PROGRAMMING PROBLEM

Industrial production, the flow of resources in the economy, the exertion of military effort in a war theater—all are complexes of numerous interrelated activities. Differences may exist in the goals to be achieved, the particular processes involved, and the magnitude of effort. Nevertheless, it is possible to abstract the underlying essential similarities in the management of these seemingly disparate systems. To do this entails a look at the structure and state of the system, and at the objective to be fulfilled, in order to *construct a statement of the actions to be performed, their timing, and their quantity (called a "program" or "schedule"), which will permit the system to move from a given status toward the defined objective.*

If the system exhibits a structure which can be represented by a mathematical equivalent, called a mathematical model, and if the objective can also be so quantified, then some computational method may be evolved for choosing the best schedule of actions among alternatives. Such use of mathematical models is termed mathematical programming. The observation that a number of military, economic, and industrial problems can be expressed (or reasonably approximated) by mathematical systems of linear inequalities and equations[1] has helped give rise to the development of linear programming.

The following three examples are typical programming problems which can be formulated linearly; they are analogous to the ones which originated research in this area [Wood and Dantzig, 1949-1; Dantzig, 1949-1]. It is well to have them in mind before we discuss the general characteristics of linear programming problems.

The objective of the system in each of the three examples to be considered happens to be the minimization of total costs measured in monetary units. In other applications, however, it could be to minimize direct labor costs or to maximize the number of assembled parts or to maximize the number of trained students with a specified percentage distribution of skills, etc.

1. A cannery example. Suppose that the three canneries of a distributor are located in Portland (Maine), Seattle, and San Diego. The canneries can fill 250, 500, and 750 cases of tins per day, respectively. The distributor operates five warehouses around the country, in New York, Chicago, Kansas City, Dallas, and San Francisco. Each of the warehouses can sell 300 cases per day. The distributor wishes to determine the number of cases to be shipped from the three canneries to the five warehouses so that each warehouse should obtain as many cases as it can sell daily at the minimum total transportation cost.

The problem is characterized by the fifteen possible *activities* of shipping cases from each of the canneries to each of the warehouses (Fig. 1-2-I). There are fifteen *unknown activity levels* (to be determined) which are the *amounts* to be shipped along the fifteen routes. This *shipping schedule* is generally referred to as the *program.* There are a number of constraints that a shipping schedule must satisfy to be feasible: namely, the schedule must show that each warehouse will receive the required number of cases

[1] The reader should especially note we have used the word *inequalities.* Systems of linear inequalities are quite general; linear inequality relations such as $x \geq 0$, $x + y \leq 7$ can be used to express a variety of common restrictions, such as quantities purchased, x, must not be negative or the total amount of purchases, $x + y$, must not exceed 7, etc.

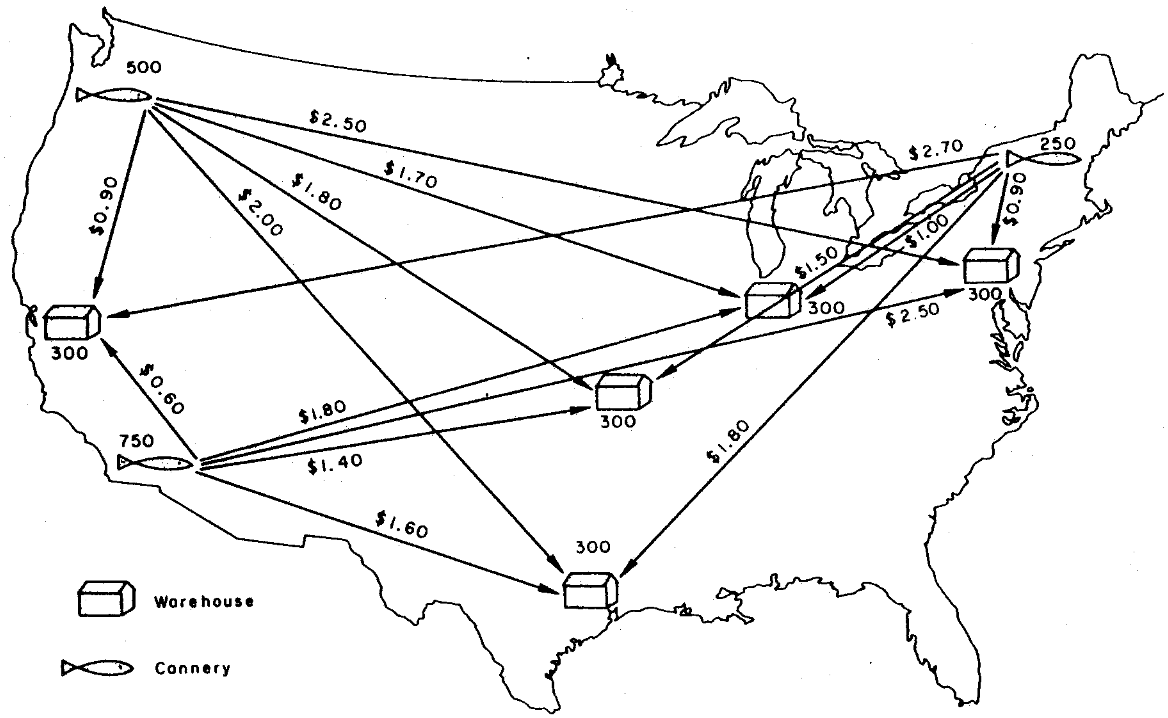


Figure 1-2-I. The Problem: Find a least cost plan of shipping from canneries to warehouses (the costs per case, availabilities and requirements are as indicated).

and that no cannery will ship more cases than it can produce daily. (Note there is one constraint for each warehouse and one for each cannery.) Several *feasible shipping schedules* may exist which would satisfy these constraints, but some will involve larger shipping costs than others. The problem then is to determine an *optimal shipping schedule*—one that has least costs. *Transportation* problems such as this are formulated in mathematical terms in § 3-3 and their solution properties are studied in Chapters 14 to 20.

2. The housewife's problem. A family of five lives on the modest salary of the head of the household. A constant problem is to determine the weekly menu after due consideration of the needs and tastes of the family and the prices of foods. The husband must have 3,000 calories per day, the wife is on a 1,500-calorie reducing diet, and the children require 3,000, 2,700, and 2,500 calories per day, respectively. According to the prescription of the family doctor, these calories must be obtained for each member by eating not more than a certain amount of fats and carbohydrates and not less than a certain amount of proteins. The diet, in fact, places emphasis on proteins. In addition, each member of the household must satisfy his or her daily vitamin needs. The problem is to assemble menus, one for each week, that will minimize costs according to Thursday food prices.

This is a typical linear programming problem: the possible activities are the purchasing of foods of different types; the program is the amounts of different foods to be purchased; the constraints on the problem are the calorie and vitamin requirements of the household, and the upper or lower limits set by the physician on the amounts of carbohydrates, proteins, and fats to be consumed by each person. The number of food combinations which satisfy these constraints is very large. However, some of these feasible programs have higher costs than others. The problem is to find a combination that minimizes the total expense[2] [Stigler, 1945-1]. *Blending* problems such as this are formulated in § 3-4.

3. On-the-job training. A manufacturing plant is contracting to make some commodity. Its present work force is considerably smaller than the one needed to produce the commodity within a specified schedule of different amounts to be delivered each week for several weeks hence. Additional workers must, therefore, be hired, trained, and put to work. The present force can either work and produce at some rate of output, or it can train some fixed number of new workers, or it can do both at the same time according to some fixed rate of exchange between output and the number of new workers trained. Even were the crew to spend one entire week training new workers, it would be unable to train the required number.

[2] Chapter 27 contains a detailed discussion of a typical nutrition problem. The reader may wonder why this problem is not really five separate problems, one for each member of the family; however, certain foods (such as eggs, milk, meat) can be subdivided into parts of varying fat content and given to different members.

The next week, the old crew *and* the newly trained workers may either work or train new workers, or may both work and train, and so on. The commodity is semi-perishable so that amounts produced before they are needed will have to be stored at a specified cost. The problem is to determine the hiring, production, and storage program that will minimize total costs.

This, too, is a linear programming problem, although with the special property, not shared with the previous two examples, of *scheduling activities through time*. The activities in this problem are the assignment of old workers to either of two jobs, production or training, and the hiring of new workers each week. The quantities of these activities are restricted by the number of workers available at the beginning of each week and by the instructor-student ratio. The cumulative output produced by all workers through the number of weeks in the contractual period has to equal or exceed the required output. A possible production-training program is shown in Fig. 1-2-II. The problem can now be stated more precisely: determine the proper balance between hiring and training of workers, between teaching and production, and between over- and under-production in order to minimize

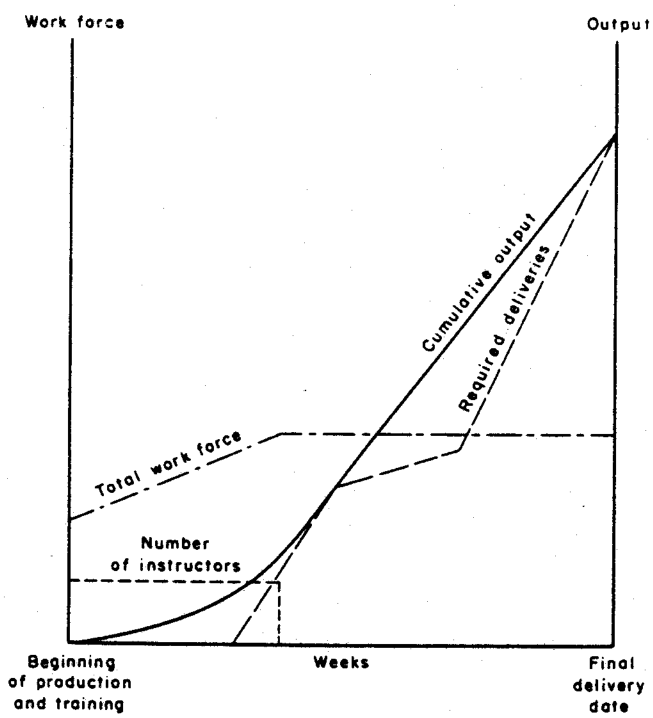


Figure 1-2-II. The Problem: Determine a least-cost hiring, production and storage program to meet required deliveries.

total costs. The mathematical formulation of this problem will be found in § 3-7.

1-3. LINEAR PROGRAMMING DEFINED

We shall use the term *model building* to express the process of putting together of symbols representing objects according to certain rules, to form a structure, *the model*, which corresponds to a system under study in the real world. The symbols may be small-scale replicas of bricks and girders or they may be, as in our application, algebraic symbols.

Linear programming has a certain philosophy or approach to building a model that has application to a broad class of decision problems encountered in government, industry, economics, and engineering. It probably possesses the simplest mathematical structure which can be used to solve the practical scheduling problems associated with these areas. Because it is a method for studying the behavior of systems, it exemplifies the distinguishing feature of management science, or operations research, to wit: "Operations are considered as an entity. The subject matter studied is not the equipment used, nor the morale of the participants, nor the physical properties of the output, it is the combination of these in total as an economic process" [Herrmann and Magee, 1953-1].

Linear programming[3] is concerned with describing the interrelations of the components of a system. As we shall see, the first step consists in regarding a system under design as composed of a number of elementary functions that are called "activities."[4] As a consequence, T. C. Koopmans [1951-1] introduced the term *activity analysis* to describe this approach. The different activities in which a system can engage constitute its technology. These are the representative building blocks of different types that might be recombined in varying amounts to rear a structure that is self-supporting, satisfies certain restrictions, and attains as well as possible a stated objective. Representing this structure in mathematical terms (as we shall see in Chapter 3) often results in a system of linear inequalities and equations; when this is so, it is called a linear programming model. Like architects, people who use linear programming models manipulate "on paper" the symbolic representations of the building blocks (activities) until a satisfactory design is obtained. The theory of linear programming is concerned with scientific procedures for arriving at the best design, given the technology, the required specifications, and the stated objective.

To be a linear programming model, the system must satisfy certain assumptions

[3] The term "linear programming" was suggested to the author by T. C. Koopmans in 1951 as an alternative to the earlier form, "programming in a linear structure" [Dantzig, 1948-1].

[4] The term "activity" in this connection is military in origin. It has been adopted in preference to the term "process," used by von Neumann in "A Model of General Economic Equilibrium," which is more restricted in connotation [von Neumann, 1937-1].

of proportionality, nonnegativity, and additivity. How this comes about will be the subject of Chapter 3, where we shall also formulate linear programming models for examples like those already discussed. It is important to realize in trying to construct models of real-life situations, that life seldom, if ever, presents a clearly defined linear programming problem, and that simplification and neglect of certain characteristics of reality are as necessary in the application of linear programming as they are in the use of any scientific tool in problem solving.

The rule is to *neglect the negligible.* In the cannery example, for instance, the number of cases shipped and the number received may well differ because of accidental shipping losses. This difference is not known in advance and may be unimportant. In the optimum diet example the true nutritional value of each type of food differs from unit to unit, from season to season, from one source of food to another. Likewise, production rates and teaching quality will vary from one worker to another and from one hour to another. In some applications it may be necessary to give considerable thought to the differences between reality and its representation as a mathematical model to be snre that the differences are reasonably small and to assure ourselves that the computational results will be operationally useful.

What constitutes the proper simplification, however, is subject to individual judgment and experience. People often disagree on the adequacy of a certain model to describe the situation.

1-4. CLASSIFICATION OF PROGRAMMING PROBLEMS

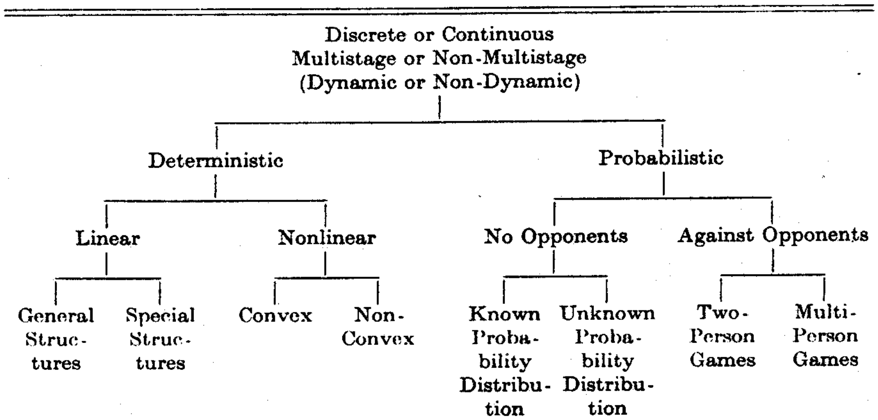
The programming problems treated in this book, except those of Chapter 25, belong to the *deterministic* class, by which it is meant that if certain actions are taken it can be predicted with *certainty* what will be (a) the requirements to carry out the actions and (b) the outcome of any actions. Few, if any, activities of the real world have this property. Perhaps the activity of burning two parts of hydrogen to one part oxygen to produce water might be cited as a deterministic example. In practice, however, because of contamination, leaks in containers, etc., this assumed relation is an ideal. For many purposes, however, an ideal formula can be used because the deviations from it are so slight that only small adjustments will be necessary from time to time.

A deterministic situation may be created by fiat. For example, the amounts of gas and oil required to carry out certain transportation activities by trucks can never be known with certainty. However, if stocks well above known expected values are used in the plan, it can be assumed that the transportation can be accomplished and any surplus stocks remaining put to good use later. Usually the working time it takes to accomplish a task is a fraction of the time assumed in the plan. For example, consider the fabrication of a part for an airplane: the elapsed time from when it is first cut out of sheet metal until it is ready to be assembled on the airplane may be three months or more; on the other hand, only a few hours may be spent shaping, drilling holes, and mounting. The remainder of the time is accounted for in storage bins whose principal function appears to be that of piling up work orders so that they can be redistributed in such a way that workers can be more effectively employed.

Programs involving *uncertainty* form the other major class which we will call *probabilistic*. Uncertainty can arise in many ways. The outcome of a given action may depend on some chance event such as the weather, traffic delays, government policy, employment levels, or the rise and fall of customer demand. Sometimes the distribution of the chance events is known, sometimes it is unknown or partially known. In some cases uncertainty arises because of the actions of competitors or enemies.

In Fig. 1-4-I the two main classes of programming problems are deterministic and probabilistic. The former is shown subdivided into two main

CLASSIFICATION OF PROGRAMMING PROBLEMS



SPECIAL CASES OF THE ABOVE

Linear inequality theory	Dynamic systems, Leontief models, Networks	Decreasing payoff, Chemical equilib- rium, Convex programs	Increasing returns to scale, Many local maxima	Inventory control, Markov chains	Sequential decisions	Zero-sum games	Coalition theory

Figure 1-4-I.

mathematical classes that are often studied, linear and nonlinear models; the latter is shown subdivided into two main application classes, those involving an indifferent (or unpredictable) nature, and those against an unfriendly opponent. These are further subdivided and well-known special cases are shown directly below each class.

In this book we will pay particular attention to both general and special linear programming structures, to nonlinear convex programming problems that can be reduced to linear programming problems, and to certain probabilistic problems that can also be reduced to linear programming problems, such as two person-zero sum games, and scheduling problems involving uncertain demand.

One important way to classify programming problems is into multistage and non-multistage groups. *Multistage models* include dynamic models in which the schedule over time is a dominant feature, as in example (3). Examples (1) and (2) are non-multistage problems as are steady-state economic models (whose production rates remain constant over time).

A second important way to classify models is into those in which some of the inputs, outputs, assignments, or production levels to be determined must occur in *discrete* amounts such as 0, 1, 2, . . . (with no intermediate amounts possible), and into those in which these quantities can take on any values over *continuous* ranges. Many combinatorial problems belong to the discrete class, such as problems concerning the assignment of a number of men to an equal number of jobs or the order in which a salesman should visit a number of cities. Strictly speaking, the discrete problems belong to the class of nonlinear programming problems (see Chapter 26).

Dynamic Programming.

Many multistage problems, particularly dynamic problems, exhibit a structure that permits a solution by application of an *inductive principle*. At the beginning of each stage, as in a treasure hunt, directions are given where to proceed next; and the total payoff of future actions, if one continues to follow directions, is indicated. It is assumed (and this is the fundamental assumption on structure) that the optimal direction and payoff depend only on one's *status* at the beginning of a stage, and not on any previous action. At the end of the *last stage* it is usually easy to give the value for all possible final states. This permits one to construct, without too much effort, the direction for maximum payoff from each of the *possible states* at the end of the *next to the last* stage; and from that, to construct the directions for maximum payoff for all possible states at the end of the *second to last* stage, and so forth *inductively backwards in time* until the beginning of the first stage where initial status is assumed known. To proceed backward in this manner it is necessary to know, for every combination of states at the beginning and end of a stage, the gain or loss within a stage. Whether or not the method can be used depends on whether the analysis of the possible combinations is tractable. The inductive principle is as old as the Greeks, but in connection with its early application to decision problems the names of A. Wald [1950-1], P. Massé [1946-1], K. Arrow, T. Harris, and

J. Marschak [1951-1], A. Dvoretsky, J. Kiefer, and J. Wolfowitz [1952-1] are worthy of mention. Richard Bellman in 1952, however, was the first to see the importance of the inductive principle which he calls the *principle of optimality* to programming applications and has been active in developing its potentialities [Bellman, 1954-1 and 1957-1; Bellman, Glicksberg, and Gross, 1958-1]. The general area of research using this principle is called "Dynamic Programming" because most of its applications happen to be multistage in character.

1-5. MATHEMATICAL PROGRAMMING AND AUTOMATION

The period following World War II has been marked by an accelerated trend toward automation, an advanced form of mechanization. Mechanization's effect is to relieve man of the need to use his human energy for power; automation's effect is to relieve him of certain of his mental tasks and the related necessary physical tasks. Many believe that electronic computers, which are themselves examples of automation, will play an important role in the mechanization of control processes of the routine type.

It is believed by some that "higher level decisions will be made by man primarily because he, through the exercise of his mind, possesses the only means of integrating and interrelating data for which rational formulations are not yet possible or are too expensive" [Boelter, 1955-1]. However, the author believes that even in the realm of higher order controls, particularly the mental tasks which involve choice of selection among alternative courses of action, mechanization is in progress. This applies to mental tasks, known as programming (or scheduling), and their physical realization, known as production control.

These two postwar developments, automation and programming, are often associated because of their use of electronic computers. How closely are they related?

To answer the question, let us inspect some developments in an industry which was one of the first to automatize production and to introduce the programming of the production process. Production in a modern petroleum refinery is a complex of interrelated activities. The number of possible combinations of feed stocks, operating sequences, operating conditions, blending methods, and the choice of final products, is large; as a consequence, mathematical programming methods are used to great advantage in evaluating the economy of an operational scheme. Once the proper production schedule is determined, it is only necessary to set dials and push buttons in the control rooms for the refinery to be able to deliver the products in the preassigned amounts.

This example shows that the two processes, decision-making and production control, could each become completely automated and yet could be linked by human operators who transmit the instructions from one system, the decision-making system, to the other, the production system. It should be emphasized, then, that although programming constitutes a higher order control, it is *not* equivalent to the feedback device which holds the temperature in a boiler constant. It is rather a method for deciding what that temperature should be and for how long, in order that the objective of the production may be attained.

While the mechanization of the higher order decision-making process does not always require the mechanization of the physical links by which the decisions are implemented, it is conceivable that in certain applications it may become economical to combine the two automated processes into one. Such "super-automated" processes are necessary in fast-flying rockets which require tight control and the use of flexible programming techniques. Some industries, such as the aircraft industry, are turning to multipurpose machines which can produce a variety of items depending on the settings of controls. These, in turn, can be changed by an automatic, higher-order control. Ultimately in such systems, machine failures, item rejects, and new orders may make it necessary to reprogram work loads rapidly. Here again, tight methods of production control may have to be linked mechanically to flexible automatic programming techniques.

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CHAPTER 2

ORIGINS AND INFLUENCES

In the ten years since its conception in 1947 in connection with planning activities of the military, linear programming has come into wide use in industry. In academic circles, mathematicians and economists have written books on the subject. The purpose of this chapter is to give a brief account of its origins and of the influences which brought about this rapid development. Table 2-1-I summarizes these, as well as the later growth of linear programming. Arrows indicate the direct influence of one happening on another. Interestingly enough, in spite of its wide applicability to everyday problems, linear programming was unknown before 1947. Fourier may have been aware of its potential in 1823. In the U.S.S.R. in 1939, Kantorovich made proposals that were neglected during the two decades that witnessed the discovery of linear programming and its firm establishment elsewhere.

2-1. WORLD WAR II INFLUENCES

The Nature of Staff Planning.

A nation's military establishment, in wartime or in peace, is a complex of economic and military activities requiring almost unbelievably careful coordination in the implementation of plans produced in its many departments. If one such plan calls for equipment to be designed and produced, then the rate of ordering equipment has to be coordinated with the capabilities of the economy to relinquish men, material, and productive capacity from the civilian to the military sector. These development and support activities should dovetail into the military program itself. To give some idea of the interdependence of various major activities there are hundreds of subtypes within each of its major activities for the case of personnel, and thousands of subtypes for the case of supply. Was it always so complicated? The following statement of M. K. Wood and M. A. Geisler [1951-1, p. 189] is pertinent:

"It was once possible for a Supreme Commander to plan operations personally. As the planning problem expanded in space, time, and general complexity, however, the inherent limitations in the capacity of any one man were encountered. Military histories are filled with instances of commanders who failed because they bogged down in details, not because they could not eventually have mastered the details, but because they could not master all the relevant details in the time available for decision. Gradually, as planning problems became more complex, the Supreme Commander came to be surrounded with a General Staff of specialists, which supplemented the Chief in making decisions. The existence of a General Staff permitted the subdivision of the planning process and the assignment of experts to handle each part. The function of the Chief then became one of selecting objectives, coordinating, planning, and resolving conflicts between staff sections."

TABLE 2-1-I

LINEAR PROGRAMMING TIMETABLE: ORIGINS—INFLUENCES

Military	Economic—Industrial		Mathematical	
	Application	*Theory*	*Theory*	*Computations*
Supreme Commander plans operations personally			Fourier (1823) Gauss (1826)	Fingers Abacus (600 B.C.)
Emergence of the General Staff (1860)	*Tableau Economique* (Quesnay, 1759)	Walrasian System (1874)	Gordan (1873) Minkowski (1896) Farkas (1903) Stiemke (1915) Statistical Theory (1900) (Neyman, 1936; Wald, Dantzig, 1939)	Pascal's Adding Machine (1642) Babbage's Difference Engine (1822)
Subdivision of Programming Process among Staff Agencies (1917)	Government Studies on Large Scale (1932)			Hollerith's Punch Card Machine (1890)
Central Monitoring of the Programming Process (1942)	Leontief Input-Output Model (1936)	von Neumann Equilibrium Model (1937)	Motzkin (1936) Kantorovich (1939) Hitchcock Transportation Problem (1941)	Relay Computers (Aiken, 1946; Stibitz, Williams)
	Scheduling War Production (1940)	Koopmans Model of Transportation (1944)	von Neumann-Morgenstern Game Theory (1944)	ENIAC Electronic Computer (von Neumann, Mauchley, Eckert, Goldstine, 1945)
Mechanization (1947) Linear Programming (1947) Triangular Model (1948) (Wood, Dantzig, Geisler) Air Force Codes BINAC 1949 IBM-604 1949 SEAC 1950 UNIVAC 1952		Koopmans, General Economic Model (1948) Dorfman's *Theory of the Firm* (1951)	Simplex Method (Dantzig, 1947) Duality (von Neumann, 1947) Theory (Tucker, Gale, Kuhn, Koopmans, 1948)	Simplex Solution Nutrition Problem (Laderman, 1948) Transportation Problem (Dantzig, 1949)
L.P. Applications: Contract Bidding (1952) Scheduling, Training, Deployment (1952) Personnel Assignment (1952) Airlift Routing Manne (1954)	*Industry* Refinery Scheduling Charnes, Cooper, Mellon (1951) Food Distribution (1953) Metal Working Industry (Markowitz, 1953) Paper Mills (1954) Electric Power (Massé, Gibrat, 1955) Numerous Applications (after 1955), many similar to 1939 proposals of Kantorovich	Dorfman, Samuelson, Solow's, "L.P. and Economic Analysis" (1958)	Chicago Conference (1949) Washington Symposia on L.P. (1951, 1955) USSR Developments (1957) Integer Programming Traveling Salesman (Dantzig, Fulkerson, Johnson, 1954) General Theory (Gomory, 1958) Uncertainty (Dantzig, 1954) Network Theory (Orden, 1950; Kuhn; Ford-Fulkerson, 1956) Quadratic, Nonlinear (Markowitz; Wolfe, 1955) Decomposition Principle (Dantzig, Wolfe, 1958) RAND Symposium (1959)	Simplex Code SEAC 1952 Orchard-Hays RAND Codes 1953–IBM CPC 1954–IBM 701 1956–IBM 704

Large wars have been waged throughout the history of civilization, but the General Staff of the Supreme Commander of military forces emerged only around the middle of the last century (Prussia, 1860) as a consequence of the increased complexity of warfare. The subdivision of the planning of military activities among the staff agencies dates back only to the stalemate and attrition phase of World War I (1917).

World War II Developments.

World War II witnessed the development of staff planning on a gigantic scale in all parts of the U.S. military establishment and in such civilian counterparts as the War Production Board. During this period the U.S. Air Corps grew to a principal arm of the military. Unfettered by tradition, it evolved a number of aids to planning[1] that ultimately led to the consideration of a scientific programming technique in the postwar period.

During the war, the planning process itself became so intricate, lengthy, and multipurposed that a snapshot of the Air Staff at any one time showed it to be working on many different programs—some in early phases of development and based on latest ground rules and status reports, others in later phases but based on earlier ground rules and facts. To cut the time of the planning process, a patchwork of several of these programs was often thrown together based on necessarily inconsistent facts and rules. To coordinate this work better, the Air Staff, around 1943, created the *program monitoring* function under Professor E. P. Learned of Harvard. The entire program was started off with a war plan in which were contained the wartime objectives. From this plan, by successive stages, the wartime program specifying unit deployment to combat theaters, training requirements of flying personnel and technical personnel, supply and maintenance, etc., was computed. To obtain consistent programming the ordering of the steps in the schedule was so arranged that the flow of information from echelon to echelon was only in one direction, and the timing of information availability was such that the portion of the program prepared at each step did not depend on any following step. Even with the most careful scheduling, it took about seven months to complete the process.

Post-World War II Developments.

After the war the U.S. Air Force consolidated the statistical control, program monitoring, and budgeting functions under the staff of the Air Force Comptroller, General E. W. Rawlings, now President of General Mills Corporation. It became clear to members of this organization that efficiently coordinating the energies of whole nations in the event of a total

[1] The most important of these was the development under C. B. Thornton of the Statistical Control System that provided a continuous flow of detailed information on the status of many parts of the Air Force, including personnel, supply, operations, and basic data upon which to base attrition rates, sortie rates, crew rotation rates, maintenance needs, supply rates, etc.

war would require scientific programming techniques. Undoubtedly this need had occurred many times in the past, but this time there were two concurrent developments that had a profound influence: (a) the development of large scale electronic computers, and (b) the development of the inter-industry model. The latter is a method of describing the inter-industry relations of an economy and was originated by Wassily Leontief [1951-1]. This is described in the next section.

Intensive work began in June 1947, in a group that later (October 1948) was given the official title of Project SCOOP (Scientific Computation of Optimum Programs). Principals in this group were Marshall Wood and the author, and soon thereafter John Norton and Murray Geisler.

The potential attraction of the inter-industry model will become apparent in the next section. Its simple structure, particularly its use of linear production functions in the description of industry-wide aggregates of economic activities, had a considerable impact on the thinking of the Air Force research team. Its nondynamic character, however, and the simplifying assumption that each industry had a unique technology which produced only one product, restricted the model's usefulness. Another limitation of the model was that it was not possible to have alternative feasible programs. It was therefore necessary to generalize the inter-industry approach. The result was the development of the linear programming model by July 1947.

The *simplex computational method* for choosing the optimal feasible program was developed by the end of the summer of 1947 (see Chapter 5). Interest in linear programming began to spread quite rapidly. During this period the Air Force sponsored work at the U.S. Bureau of Standards on electronic computers and on mathematical techniques for solving such models. John Curtiss and Albert Cahn of the Bureau played an active role in generating interest in the work among economists and mathematicians.

Contact with Tjalling Koopmans of the Cowles Commission, then at the University of Chicago, now at Yale, and Robert Dorfman, then of the Air Force, now at Harvard, and the interest of such economists as Paul Samuelson of the Massachusetts Institute of Technology, initiated an era of intense re-examination of classical economic theory using results and ideas of linear programming.

Contact with John von Neumann at the Institute for Advanced Study gave fundamental insight into the mathematical theory and sparked the interest of A. W. Tucker of Princeton University and a group of his students, who attacked problems in linear inequality theory and game theory. Since that time his group has been a focal point of work in these related fields.

It was the size of the Air Force programming problem which made the SCOOP personnel recognize, at an early date, that even the best of future computing facilities would not be powerful enough to solve a general detailed Air Force linear programming model. Accordingly, Project SCOOP modified its approach and in the spring of 1948 proposed that there be developed special linear programming models called *triangular models* whose structure and computational solution would parallel the stepwise staff procedure which we described earlier [Wood and Geisler, 1951-1, p. 189].

Since 1948 the Air Staff has been making more and more active use of mechanically computed programs. The triangular models are in constant use for the computation of detailed programs, while the general linear programming models have been applied in certain areas, such as (a) contract bidding, (b) balanced aircraft, crew training, and wing deployment schedules, (c) scheduling of maintenance overhaul cycles, (d) personnel assignment, and (e) airlift routing problems [U.S. Air Force, 1954-1; Jacobs, 1955-1; Natrella, 1955-1].

2-2. ECONOMIC MODELS AND LINEAR PROGRAMMING

The Influence of Theoretical Models.

The current introduction of linear programming in economics appears to be an anachronism; it would seem logical that it should have begun around 1758 when economists first began to describe economic systems in mathematical terms. Indeed, a crude example of a linear programming model can be found in the *Tableau économique* of Quesnay, who attempted to interrelate the roles of the landlord, the peasant, and the artisan [Monroe, 1924-1]. Also, we find that L. Walras proposed in 1874 a sophisticated mathematical model which had as part of its structure fixed technological coefficients. Oddly enough, however, until the 1930's there was little in the way of exploitation of the linear-type model.

For the most part, mathematical economists were occupied with the analysis of theoretical problems associated with the possibility of economic equilibria and its allocative efficiency under competitive or monopolistic conditions. For such studies they found the use of classical convex functions with continuous derivatives more convenient for the demonstration of stability conditions than functions based on linear inequalities. Of particular note, along these lines, is the effort during the 1930's of a group of Austrian and German economists who worked on generalizations of the linear technology of Walras. This work raised some questions that may have stimulated the mathematician von Neumann (1932), in his paper "A Model of General Economic Equilibrium" [von Neumann, 1937-1], to formulate a dynamic linear programming model in which he introduced alternative methods of producing given commodities singly or jointly. Von Neumann assumed (a) a constant rate of expansion of the economy, and (b) a completely self-supporting economy. While the model did not contain any explicit objective, von Neumann showed that market forces would maximize the expansion rate, and proved that at the maximum it was equal to the interest rate on capital invested in production.

As far as influence is concerned, von Neumann's paper, like many other theoretical papers, proved only an interesting mathematical theorem. It is likely that mathematical economists were more interested in getting similar results for a more general model because "To many economists the term linearity is associated with narrowness, restrictiveness, and inflexibility of hypotheses" [Koopmans, 1951-1, p. 6]. In other words, this effort belonged like many others to the qualitative world of the economics of that time, a world in which the purpose of the mathematical model was to describe in a *qualitative* rather than a *quantitative* way the assumed interrelations within a system; the manipulation of equations was a convenient way to make valid logical deductions from the assumptions.

The Influence of Empirical Models.

The inspiration of the general linear programming model was completely independent of these developments and had a different purpose. It arose out of the empirical programming needs of the Air Force and the possibility of generalizing the simple practical structure of the Leontief Model to this end. From a purely formal standpoint the Leontief Model can be considered as a simplification of the Walrasian Model. It is here that the formalism ends.

"One hundred and fifty years ago, when Quesnay first published his famous schema, his contemporaries and disciples acclaimed it as the greatest discovery since Newton's laws. The idea of general interdependence among the various parts of the economic system has become by now the very foundation of economic analysis. Yet, when it comes to the practical application of this theoretical tool, modern economists must rely exactly as Quesnay did upon fictitious numerical examples" [Leontief, 1951-1, p. 9].

Leontief's great contribution, in the opinion of the author, was his construction of a *quantitative model* of the American economy, for the purpose of tracing the impact of government policy and consumer trends upon a large number of industries which were imbedded in a highly complex series of interlocking relationships. To appreciate the difference between a purely formal model and an empirical model, it is well to remember that the acquisition of data for a real model requires an organization working many months, sometimes years. After the model has been put together, another obstacle looms—the solution of a very large system of simultaneous linear equations. In the period 1936–1940, there were no electronic computers; the best that one could hope for in general would be to solve twenty equations in twenty unknowns. Finally, there was the difficulty of "marketing" the results of such studies. Hence, from the onset, the undertaking initiated by Leontief represented a triple gamble.

To appreciate further the significance of this shift from the theoretical to the empirical model it should be remembered that since the 1930's much more information has become available on income, quantities of production, investment, savings, and consumer patterns. Moreover, since 1900, sampling techniques developed by statisticians have come more and more into use as a means of evaluating the interrelationships between observations. Regression analysis began to be used to measure economic phenomena. By 1940 the work of such statisticians as Karl Pearson, R. A. Fisher, and the modern school initiated by J. Neyman had become a science for testing hypotheses and evaluating the parameters in the statistical population.

As a result of the great depression and the advent of the "New Deal" there was a serious attempt on the part of the government to determine, and then support, certain activities which it was hoped would speed recovery. This brought about more intensive collections of statistics on costs of living, wages, national resources, productivity, etc. There was a need to organize and interpret this data by using it to construct a mathematical model to describe the economy in quantitative terms.

From 1936 on, the scope, accuracy, and area of application of Leontief-type models were greatly extended by the Bureau of Labor Statistics (under the direction of Duane Evans, Jerome Cornfield, Marvin Hoffenberg, and others) [Cornfield, Evans, and Hoffenberg, 1947-1]. It was this work that stimulated efforts toward seeking a mathematical generalization suitable for dynamic Air Force applications. Thus the early Air Force interest was in the mathematical structure; it was not until several years later that the military supported work on Leontief inter-industry models to help evaluate the interaction of their programs with the civilian economy.

A few words about the Leontief model itself are in order. The focal point of input-output analysis is an array of coefficients variously called the "input-output" matrix or "tableau économique." A *column* of this matrix represents the input requirements of various commodities for the production of one dollar's worth of a particular commodity. There is exactly one column for each commodity produced in the economy. Thus the *production of a commodity* corresponds to the concept of an *activity* in a linear programming model. If the input factors appearing in a *row* of the matrix are multiplied by the corresponding buying industry's total output, the totals represent the distribution of the dollar value of purchases among the selling industries. Thus, the model makes it possible not only to determine each industry's rate of output to meet specified direct demand by civilians and the military, but also to trace the indirect effect on each industry of government expenditures in, say, military programs.

Postwar Developments.

In 1947, T. C. Koopmans took the lead in bringing to the attention of economists the potentialities of the linear programming models. His rapid development of the economic theory of such models was due to the insight he gained during the war with a special class of linear programming models called *transportation models*. He organized the historic Cowles Commission conference on "linear programming," referred to in Chapter 1. At the conference were such well-known economists as K. Arrow, R. Dorfman, N. Georgescu-Roegen, L. Hurwicz, A. Lerner, J. Marschak, O. Morgenstern, S. Reiter, P. Samuelson, and H. Simon; such mathematicians as G. W. Brown, M. M. Flood, D. Gale, H. W. Kuhn, C. B. Tompkins, A. W. Tucker, and the author, as well as government statisticians, including W. D. Evans, M. A. Geisler, M. Hoffenberg, and M. K. Wood. The papers presented there were later collected into the book *Activity Analysis of Production and Allocation* [Koopmans, 1951-1]. The book reflects the interest awakened among these groups in two short years. The following is an interesting quotation from its introduction, in which Koopmans encourages theoretical economists to set aside some of their traditional beliefs:

"The adjective in 'linear model' relates only to (a) assumption of proportionality of inputs and outputs in each elementary productive activity, and (b) the assumption that the result of simultaneously carrying out two or more activities is the sum of the results of the separate activities. In terms more familiar to the economist, these assumptions imply constant returns to scale in all parts of the technology. They do not imply linearity of the production function. . . . Curvilinear production functions . . . can be obtained from the models here studied by admitting an infinite set of elementary activities. . . .

"Neither should the assumption of constant returns to scale . . . be regarded as essential to the method of approach it illustrates, although new mathematical problems would have to be faced in the attempt to go beyond this assumption. More essential to the present approach is the introduction of . . . the elementary activity, the conceptual atom of technology into the basic postulates of the analysis. The problem of efficient production then becomes one of finding the proper rules for combining these building blocks. The term 'activity analysis' . . . is designed to express this approach" [Koopmans, 1951-1, p. 6].

Koopmans was the first to point out that many theorems of welfare economics, the study of the rules for efficient allocation of resources in the economy, could be restated under the assumption of a linear technology for the "firm." The decisions to be made by his "helmsman" on resource allocation did not conflict with earlier results of traditional economic theory; indeed, they were more general in that the decisions covered joint products and by-products of the firm [Koopmans, 1951-2].

At about the same time, a few other economists had become interested in activity analysis and linear programming. Dorfman (1951) expressed in linear programming terms the economic theory of the firm under competitive and monopolistic conditions, and compared the realm of applicability of this theory with the traditional marginal analysis [Dorfman, 1951-1]. Samuelson (1955) wrote on "Market Mechanisms and Maximization" and stated his Substitution Theorem for a Generalized Leontief Model [Samuelson, 1955-1;

Koopmans, 1951-1]. Various classical economic problems, such as international trade between two countries and the Giffen paradox, could be reformulated as linear programming problems [Beckmann, 1955-1; Dorfman, Samuelson, and Solow, 1958-1; Koopmans, 1951-1].

The number of practical economic applications is continually growing. Linear programming is being used by economists to study in detail the economics of specific industries, such as metalworking [Markowitz, 1954-1], petroleum refining,[2] iron and steel [Fabian, 1958-1], and to yield long-range plans for electricity generation in an entire economy [Massé and Gibrat, 1957-1]. Some of these applications will be presented as examples and exercises in later chapters.

For a fuller appreciation of the economic implications, the reader is referred to *Linear Programming and Economic Analysis* by Dorfman, Samuelson, and Solow [1958-1], and *Economic Theory and Operations Analysis* by W. J. Baumol [1961-1].

2-3. MATHEMATICAL ORIGINS AND DEVELOPMENTS

History Prior to 1947.

The linear programming model, when translated into purely mathematical terms, as will be done in the next chapter, requires a method for finding a solution to a system of simultaneous linear equations and linear inequalities which minimizes a linear form. This central mathematical problem of linear programming was not known to be an important one with many practical applications until the advent of linear programming in 1947. It is this which in part accounts for the lack of active interest among mathematicians in finding efficient solution techniques before that date.

We are all familiar with methods for solving linear equation systems which start with our first course in algebra [Gauss, 1826-1; Jordan, 1904-1]. The literature of mathematics contains thousands of papers concerned with techniques for solving linear equation systems, with the theory of matrix algebra (an allied topic), with linear approximation methods, etc. On the other hand, the study of linear inequality systems excited virtually no interest until the advent of game theory in 1944 and linear programming in 1947. For example T. Motzkin, in his doctoral thesis on linear inequalities in 1936, was able to cite after diligent search only some thirty references for the period 1900–1936, and about forty-two in all [Motzkin, 1936-1]. In the 1930's, four papers dealt with the building of a comprehensive theory of linear inequalities and with an appraisal of earlier works. These were by R. W. Stokes [1931-1], Dines-McCoy [1933-1], H. Weyl [1935-1], and T. Motzkin [1936-1]. As evidence that mathematicians were unaware of the

[2] [Charnes, Cooper, and Mellon, 1952-1; Symonds, 1955-1; Manne, 1956-1; Garvin, Crandall, John, and Spellman, 1957-1.]

importance of the problem of seeking a solution to an inequality system that also minimized a linear form, we may note that none of these papers made any mention of such a problem, although there had been earlier instances in the literature.

The famous mathematician, Fourier, while not going into the subject deeply, appears to have been the first to study linear inequalities systematically and to point out their importance to mechanics and probability theory [Fourier, 1826-1]. He was interested in finding the *least maximum deviation* fit to a system of linear equations, which he reduced to the problem of finding the lowest point of a polyhedral set. He suggested a solution by a vertex-to-vertex descent to a minimum, which is the principle behind the simplex method used today. This is probably the earliest known instance of a linear programming problem. Later another famous mathematician, de la Vallée Poussin [1911-1], considered the same problem and proposed a similar solution.

A good part of the early mathematical literature is concerned with finding conditions under which a general homogeneous linear inequality system can be solved. All the results obtained express, in one form or another, a relationship between the original (or *primal*) system and another system (called the *dual*) which uses the columns of the original matrix of coefficients to form new linear equations or inequalities according to certain rules. Typical is the derived theorem of P. Gordan [1873-1] showing that a homogeneous system of equations in nonnegative variables possesses a solution with at least one variable positive if the dual possesses no solution with strict inequalities. Stiemke [1915-1] added a theorem on the existence of a solution with all variables positive. These results are expressed in a sharper form in Motzkin's Transposition Theorem [1936-1] and theorems on Dual Systems by Tucker [1956-1]. Specifically designed for algebraic proof of the Minimax Theorem are the results of Ville [1938-1] and of von Neumann and Morgenstern [1944-1]. Essentially, these theorems state that either the original (primal) system possesses a nontrivial solution or the dual system possesses a strict inequality solution. Because of this "either-or," von Neumann and Morgenstern called their result the Theorem of the Alternative for Matrices (see § 6-4).

The following is a well-known theorem for equations: If every solution to a linear equation system also satisfies a given linear equation, the equation can be formed as a linear combination of the equations of the system. A surprising and important theorem for inequalities due to J. Farkas [1902-1] is as follows: If every solution to a linear homogeneous inequality system also satisfies a given linear inequality (where all inequalities are ≥ 0), the inequality can be formed as a nonnegative linear combination of the inequalities of the system.

Analogous to those for equation systems, other theorems are concerned with building up a general solution of an inequality system by forming a

[1896-1], states that for a homogeneous system the general solution can be formed as a nonnegative linear combination of a finite number of essential solutions variously called *extreme solutions*, *vertex solutions*, or *basic solutions* (as used in this text).

The Work of Kantorovich.

The Russian mathematician L. V. Kantorovich has for a number of years been interested in the application of mathematics to programming problems. He published an extensive monograph in 1939 entitled *Mathematical Methods in the Organization and Planning of Production* [1939-1].

In his introduction Kantorovich states, "There are two ways of increasing efficiency of the work of a shop, an enterprise, or a whole branch of industry. One way is by various improvements in technology, that is, new attachments for individual machines, changes in technological processes, and the discovery of new, better kinds of raw materials. The other way, thus far much less used, is by improvement in the organization of planning and production. Here are included such questions as the distribution of work among individual machines of the enterprise, or among mechanisms, orders among enterprises, the correct distribution of different kinds of raw materials, fuels, and other factors" [Kantorovich, 1939-1].

Kantorovich should be credited with being the first to recognize that certain important broad classes of production problems had well-defined mathematical structures which, he believed, were amenable to practical numerical evaluation and could be numerically solved.

In the first part of his work Kantorovich is concerned with what we now call the weighted two-index distribution problems. These were generalized first to include a single linear side condition, then a class of problems with processes having several simultaneous outputs (mathematically the latter is equivalent to a general linear program). He outlined a solution approach based on having on hand an initial feasible solution to the dual. (For the particular problems studied, the latter did not present any difficulty.) Although the dual variables were not called "prices," the general idea is that the assigned values of these "resolving multipliers" for resources in short supply can be increased to a point where it pays to shift to resources that are in surplus. Kantorovich showed on simple examples how to make the shifts to surplus resources. In general, however, *how to shift* turns out to be a linear program in itself for which no computational method was given. The report contains an outstanding collection of potential applications.

His 1942 paper "On the Translocation of Masses" [Kantorovich, 1942-1] is the forerunner of his joint paper with M. K. Gavurin on "The Application of Mathematical Methods to Problems of Freight Flow Analysis" [Kantorovich and Gavurin, 1949-1]. Here can be found a very complete theory of the transshipment problem, the relations between the primal and the dual (price) system, the use of the linear graph of the network, and the important extension to capacitated networks. Moreover, it is clear that the authors had developed considerable facility with the adjustment of freight flow patterns from nonoptimal to optimal patterns for elaborate systems of the kind commonly encountered in practice. However, again, an incomplete computational algorithm was given. It is commendable that the paper is written in a nontechnical manner, so as to encourage those responsible for routing freight to use the proposed procedures.

In 1959, twenty years after the publication of his first work, Kantorovich published a second entitled *Economic Computation of the Optimal Utilization of Resources*, a book primarily intended for economists [1959-1].

If Kantorovich's earlier efforts had been appreciated at the time they were first presented, it is possible that linear programming would be more advanced today. However, his early work in this field remained unknown both in the Soviet Union and elsewhere for nearly two decades while linear programming became a highly developed art. According to *The New York Times*, "The scholar, Professor L. V. Kantorovich, said in a debate that, Soviet economists had been inspired by a fear of mathematics that left the Soviet Union far behind the United States in applications of mathematics to economic problems. It could have been a decade ahead" [*New York Times*, 1959-1].

Direct Influences.

With the exception of the game-theoretic results due to von Neumann and to Ville, all the work just cited seems not to have had any influence on the immediate postwar developments in linear programming. Let us now turn to those that are known to have had a direct influence.

In 1936, J. Neyman and E. S. Pearson clarified the basic concepts for validating statistical tests and estimating underlying parameters of a distribution from given observations [Neyman and Pearson, 1936-1]. They used what is now the well-known Neyman-Pearson Lemma for constructing the best test of a simple hypothesis having a single alternative. For a more general class of hypotheses they showed that if a test existed satisfying a generalized form of their lemma, it would be optimal. In 1939 (and as part of his doctoral thesis, 1946), the author first showed that under very general conditions such a test always exists. This work was later published jointly with A. Wald, who independently reached the same result around 1950 [Dantzig and Wald, 1951-1]. This effort constitutes not only an early proof of one form of the important duality theorem of linear programming, but one given for an infinite (denumerable) number of variables or (through the use of integrals) a nondenumerable number of variables. These are referred to by Duffin as *infinite programs* [Duffin, 1956-1]. It is interesting to note that the conditions of the general Neyman-Pearson Lemma are in fact the *conditions that a solution to a bounded variable linear programming problem* *be optimal*. The author's research on this problem formed a background for his later research on linear programming.

Credit for laying the mathematical foundations of this field goes to John von Neumann more than to any other man (see Kuhn and Tucker, 1958-1). During his lifetime, he was generally regarded as the world's foremost mathematician. He played a leading role in many fields; atomic energy and electronic computer development are two where he had great influence. In 1944 John von Neumann and Oskar Morgenstern published their monumental work on the theory of games, a branch of mathematics that aims to analyze problems of conflict by use of models termed "games" [von Neumann and Morgenstern, 1944-1]. A theory of games was first broached in 1921 by Emile Borel and was first established in 1928 by von Neumann with his famous Minimax Theorem [Ville, 1938-1; Borel, 1953-1]. The significance of this effort for us is that game theory, like linear programming, has its mathematical foundation in linear inequality theory [Kuhn and Tucker, 1958-1].

Postwar Developments (1947-1956).

During the summer of 1947, Leonid Hurwicz, well-known econometrician associated with the Cowles Commission, worked with the author on techniques for solving linear programming problems. This effort and some suggestions of T. C. Koopmans resulted in the "Simplex Method." The obvious idea of moving along edges from one vertex of a convex polyhedron to the next (which underlies the simplex method) was rejected earlier on intuitive grounds as inefficient. In a different geometry it seemed efficient and so, fortunately, it was tested and accepted.

Von Neumann, at the first meeting with the author in October 1947, was able immediately to translate basic theorems in game theory into their equivalent statements for systems of linear inequalities [Goldman and Tucker, 1956-1]. He introduced and stressed the fundamental importance of *duality*[3] and conjectured the equivalence of games and linear programming problems [Dantzig, 1951-1; Gale, Kuhn, and Tucker, 1951-1]. Later he made several proposals for the numerical solution of linear programming and game problems [von Neumann, 1948-1, 1954-1].

A. W. Tucker's interest in game theory and linear programming began in 1948. Since that time Tucker and his former students (notably David Gale and Harold W. Kuhn) have been active in developing and systematizing the underlying mathematical theory of linear inequalities. Their main efforts, like those of a group at The RAND Corporation (notably N. C. Dalkey, M. Dresher, O. Helmer, J. C. C. McKinsey, L. S. Shapley, and

[3] D. Ray Fulkerson, in a conversation with S. Karlin, accidentally credited the simplex method to von Neumann when he meant to credit duality to him. This error subsequently appeared in the work of Karlin [1959-1] and then was repeated by Charnes and Cooper [1961-1].

J. D. Williams), have been in the related field of game theory [von Neumann, 1948-1].

The National Bureau of Standards played an important role in the development of linear programming theory. Not only did it arrange through John H. Curtiss and Albert Cahn the important initial contacts between workers in this field, but it provided for the testing of a number of computational proposals in their laboratories. In the fall of 1947, Laderman of the Mathematical Tables Project in New York computed the optimal solution of Stigler's diet problem [Stigler, 1945-1] in a test of the newly proposed simplex method. At the Institute of Numerical Analysis, Professor Theodore Motzkin, whose work on the theory of linear inequalities has been mentioned earlier, proposed several computational schemes for solving linear programming problems such as the "Relaxation Method" [Motzkin and Schoenberg, 1954-1] and the "Double Description Method" [Motzkin, Raiffa, Thompson, and Thrall, 1953-1]. Charles B. Tompkins proposed his projection method [Tompkins, 1955-1]. Alex Orden of the Air Force worked actively with the National Bureau of Standards (N.B.S.) group who prepared codes on the SEAC (National Bureau of Standards Eastern Automatic Computer) for the general simplex method and for the transportation problem. Alan J. Hoffman, with a group at the N.B.S., was instrumental in having experiments run on a number of alternative computational methods [Hoffman, Mannos, Sokolowsky, and Wiegmann, 1953-1]. He was also the first to establish that "cycling" can occur in the simplex algorithm without special provisions for avoiding degeneracy [Hoffman, 1953-1].

In June 1951 the First Symposium in Linear Programming was held in Washington under the joint auspices of the Air Force and the N.B.S. By this time, interest in linear programming was widespread in government and academic circles. A. Charnes and W. W. Cooper had just begun their pioneering work on industrial applications. Aside from this work, which will be discussed in the next section, they published numerous contributions to the theory of linear programming. Their lectures were published in *An Introduction to Linear Programming* [Charnes, Cooper, and Henderson, 1953-1]. A two-volume treatise of the work of Charnes and Cooper was published in 1961.

Computational Developments (1947-1956).

New computational techniques and variations of older techniques are continuously being developed in the United States and abroad. Aside from those mentioned above, there were early proposals by G. W. Brown and T. C. Koopmans [Brown and Koopmans, 1951-1] and a method for solving games by G. W. Brown [Brown, 1951-1]. More recently the well-known econometrician Ragnar Frisch at the University of Oslo has done extensive research work on his "Multiplex Method" [Frisch, 1957-1]. Investigations in Great Britain have been spearheaded by S. Vajda [1958-1]. There are a number of important variants of the simplex method proposed by C. Lemke [1954-1], W. Orchard-Hays [1954-1], E. M. L. Beale [1954-1], and others (see Chapter 11).

Electronic Computer Codes (1947-1956).

The special simplex method developed for the transportation problem [Dantzig, 1951-2] was first coded for the SEAC in 1950 and the general simplex method in 1951 under the general direction of A. Orden of the Air Force and A. J. Hoffman of the Bureau of Standards. In 1952, W. Orchard-Hays of The RAND Corporation worked out a simplex code for the IBM-C.P.C., and for the IBM 701 and 704 in 1954 and 1956, respectively. The latter code was remarkably flexible and solved problems of two hundred equations and a thousand or more variables in five hours or so with great accuracy [Orchard-Hays, 1955-1].

Special routines for solving the Air Force *triangular model* were first developed in 1949. In the spring of 1949, M. Montalbano of the N.B.S. built a preliminary computation system around an IBM 602-A; later a more elaborate system was built for the IBM 604. In early 1950, with C. Diehm, he prepared a simplex code for SEAC which was demonstrated at the dedication of the computer. These computational programs were recoded by the Air Force when they obtained a UNIVAC in 1952.

The use of electronic computers by business and industry has been growing by leaps and bounds. Many of the digital computers which are commercially available have had codes of the simplex technique. In addition, there has been some interest in building *analogue computers* for the sole purpose of solving linear programming problems [Ablow and Brigham, 1955-1; Pyne, 1956-1]. It is possible that such computers may provide an efficient tool for the evaluation of parametric changes in a system represented by a linear programming model and may be useful when quick solutions of linear programming problems are continuously needed, as for example in production scheduling. These computers have worked well on small problems (for example twenty variables and ten equations). Because of distortion of electric signals, it does not seem practical to design analogue computers which can handle the large *general* linear programming problems. However it does appear very worthwhile to try to develop applications of such computers to solving large-scale systems which possess *special structures.*

Extensions of Linear Programming.

If we distinguish, as indeed we must, between those types of generalizations in mathematics that have led to existence proofs and those that have led to constructive solutions of practical problems, then the period following the first decade marks the beginning of several important constructive generalizations of linear programming concepts to allied fields. These are:

(1) *Network Theory.* A remarkable property of a special class of linear programs, the transportation or the equivalent network flow problem, is that their extreme point solutions are integer valued when their constant terms are integers [G. Birkhoff, 1946-1; Dantzig, 1951-2]. This has been a key fact in an elegant theory linking certain combinatorial problems of topology with the continuous processes of network theory. The field has many contributors. Of special mention is the work of Kuhn [1955-1] using an approach of Egerváry on the problem of finding a permutation of ones in a matrix composed of zeros and ones and the related work of Ford and Fulkerson [1954-1] for network flows. For further references, see Chapters 19 and 20, [Hoffman, 1960-1; Berge, 1958-1; Ford and Fulkerson, 1960-1].

(2) *Convex Programming.* A natural extension of linear programming occurs when the linear part of the inequality constraints and the objective are replaced by convex functions. Early work centered about a quadratic objective [Dorfman, 1951-1; Barankin and Dorfman, 1958-1; Markowitz, 1956-1] and culminated in an elegant procedure developed independently by Beale [1959-1], Houthakker [1959-1], and Wolfe [1959-1] who showed how a minor variant of the simplex procedure could be used to solve such problems. Also studied early was the case where the convex objective could be separated into a nonnegative sum of terms, each convex in a single variable [Dantzig, 1956-2; Charnes and Lemke, 1954-1]. The general case has been studied in fundamental papers by Kuhn and Tucker [1950-2], and Arrow, Hurwicz, and Uzawa [1958-1]. See Chapter 24 for further references. In this book we shall attack this problem by using the decomposition principle of linear programs (Chapters 22, 23, 24). Many promising alternative approaches can be found in the literature [Rosen, 1960-1].

(3) *Integer Programming.* Important classes of nonlinear, nonconvex, discrete, combinatorial problems can be shown to be formally reducible to a linear programming type of problem, some or all of whose variables must be integer valued. By the introduction of the concept of cutting planes, linear programming methods were used to construct an optimal tour for a salesman visiting Washington, D.C., and forty-eight state capitals of the United States [Dantzig, Fulkerson, and Johnson, 1954-1]. The theory was incomplete. The foundations for a rigorous theory were first developed by Gomory [1958-1]. See Chapter 26.

(4) *Programming under Uncertainty.* It has been pointed out by Madansky [1960-1] that the area of programming under uncertainty cannot be usefully stated as a single problem. One important class considered in this book is a multistage class where the technological matrix of input-output coefficients is assumed known, the values of the constant terms are uncertain, but the joint probability distribution of their possible values is assumed to be known. Some tools for attacking this class of problems will be found in Chapters 25 and 28. A promising approach based on the decomposition principle has been discussed by Dantzig and Madansky [1960-1].

2-4. INDUSTRIAL APPLICATIONS OF LINEAR PROGRAMMING

The history of the first years of linear programming would be incomplete without a brief survey of its use in business and industry. These applications began in 1951 but have had such a remarkable growth in the years 1955–1960 that this use is now more important than its military predecessor.

Linear programming has been serving industrial users in several ways. First, it has provided a *novel view of operations*; second, it induced *research in the mathematical analysis of the structure of industrial systems*; and third, it has become an important tool for business and industrial management for *improving the efficiency of their operations*. Thus the application of linear programming to a business or industrial problem has required the mathematical formulation of the problem and an explicit statement of the desired objectives. In many instances such rigorous thinking about business problems has clarified aspects of management decision-making which previously had remained hidden in a haze of verbal arguments. As a partial consequence some industrial firms have started educational programs for their managerial personnel in which the importance of the definition of objectives and constraints on business policies is being emphasized. Moreover, scheduling industrial production traditionally has been, as in the military, based on intuition and experience, a few rules, and the use of visual aids. Linear programming has induced extensive research in developing quantitative models of industrial systems for the purpose of scheduling production. Of course many complicated systems have not as yet been quantified, but sketches of conceptual models have stimulated widespread interest. An example of this is in the scheduling of job-shop production, where M. E. Salveson [1953-1] initiated research work with a linear programming-type tentative model. Research on job-shop scheduling is now being performed by several academic and industrial research groups [Jackson, 1957-1]. Savings by business and industry through the use of linear programming for planning and scheduling operations are occasionally reported [Dantzig, 1957-1].

The first and most fruitful industrial applications of linear programming have been to the scheduling of petroleum refineries. As noted earlier, Charnes, Cooper, and Mellon started their pioneering work in this field in 1951 [Charnes, Cooper, and Mellon, 1952-1]. Two books have been written on the subject, one by Gifford Symonds [Symonds, 1955-1] and another by Alan Manne [Manne, 1956-1]. So intense has been the development that a survey by Garvin, Crandall, John, and Spellman [1957-1] showed that there are applications by the oil industry in exploration and production and distribution as well as in refining. The routing of tanker ships by linear programming methods may soon be added to this list.

The food processing industry is perhaps the second most active user of linear programming. In 1953 a major producer first used it to determine shipping of catchup from six plants to seventy warehouses [Henderson and Schlaifer, 1954-1] and a milk producer has considered applying it to a similar problem, except that in this case the number of warehouses is several hundred. A major meat packer determines by means of linear programming the most economical mixture of animal feeds [Fisher and Schruben, 1953-1].

In the iron and steel industry, linear programming has been used for the evaluation of various iron ores and of the pelletization of low-grade ores [Fabian, 1954-1]. Additions to coke ovens and shop loading of rolling mills have provided additional applications [Fabian, 1955-1]; a linear programming model of an integrated steel mill is being developed [Fabian, 1958-1]. It is reported that the British steel industry has used linear programming to decide what products their rolling mills should make in order to maximize profit.

Metalworking industries use linear programming for shop loading [Morin, 1955-1] and for determining the choice between producing and buying a part [Lewis, 1955-1; Maynard, 1955-1]. Paper mills use it to decrease the amount of trim losses [Eisemann, 1957-1; Land and Doig, 1957-1; Paull and Walter, 1955-1; Doig and Belz, 1956-1].

The optimal routing of messages in a communication network [Kalaba and Juncosa, 1956-1], contract award problems [Goldstein, 1952-1; Gainen, 1955-1], and the routing of aircraft and ships [Dantzig and Fulkerson, 1954-1; Ferguson and Dantzig, 1954-1, 1956-1] are problems that have been considered for application of linear programming methods by the military and are under consideration by industry. In France the best program of investment in electric power has been investigated by linear programming methods [Massé and Gibrat, 1957-1].

Since 1957 the number of applications has grown so rapidly that it is not possible to give an adequate treatment here.

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CHAPTER 3

FORMULATING A LINEAR PROGRAMMING MODEL[1]

3-1. BASIC CONCEPTS

Suppose that the system under study (which may be one actually in existence, or one which we wish to design) is a complex of machines, people, facilities, and supplies. It has certain over-all reasons for its existence. For the military it may be to provide a striking force, or for industry it may be to produce certain types of products.

The linear programming approach is to consider a system as decomposable into a number of elementary functions, the *activities*. An activity is thought of as a kind of "black box"[2] into which flow tangible inputs, such as men, material, and equipment, and out of which may flow the products of manufacture, or the trained crews of the military. What happens to the inputs inside the "box" is the concern of the engineer or of the educator; to the programmer, only the rates of flow into and out of the activity are of interest. The various kinds of flow are called *items*.

The quantity of each activity is called the *activity level*. To change the activity level it is necessary to change the flows into and out of the activity.

Assumption 1: Proportionality.

In the linear programming model the quantities of flow of various items into and out of the activity are always proportional to the activity level. If we wish to double the activity level, we simply double all the corresponding flows for the unit activity level. For instance, in § 1-2, Example 3, if we wish to double the number of workers trained in a period, we would have to double the number of instructors for that period and the number of workers hired. This characteristic of the linear programming model is known as the proportionality assumption.

Assumption 2: Nonnegativity.

While any positive multiple of an activity is possible, negative quantities of activities are not possible. For example, in § 1-2, Example 1, a negative number of cases cannot be shipped. Another example occurs in a well-known classic: the Mad Hatter, you may recall, in *Alice's Adventures in Wonderland*,

[1] This chapter, written by Philip Wolfe, is based on earlier drafts by the author.
[2] Black box: Any system whose detailed internal nature one willfully ignores.

was urging Alice to have some more tea, and Alice was objecting that she couldn't see how she could take more when she hadn't had any. "You mean, you don't see how you can take *less* tea," said the Hatter, "it is very easy to take more than nothing." Lewis Carroll's point was probably lost on his pre-linear-programming audience, for why should one emphasize the obvious fact that the activity of "taking tea" cannot be done in negative quantity? Perhaps it was Carroll's way of saying that mathematicians had been so busy for centuries extending the number system from integers, to fractions, to negative, to imaginary numbers, that they had given little thought on how to keep the variables of their problems in their original nonnegative range. This characteristic of the variables of the linear programming model is known as the nonnegativity assumption.

Assumption 3: Additivity.

The next step in building a model is to specify that the system of activities be complete in the sense that a complete accounting by activity can be made of each item. To be precise, for each item it is required that the total amount specified by the system as a whole equals the sum of the amounts flowing into the various activities minus the sum of the amounts flowing out. Thus, each item, in our abstract system, is characterized by a *material balance equation*, the various terms of which represent the flows into or out of the various activities. In the cannery example, the number of cases sent into a warehouse must be completely accounted for by the amounts flowing out of the shipping activities from various canneries including possible storage or disposal of any excess. This characteristic of the linear programming model is known as the additivity assumption.

Assumption 4: Linear Objective Function.

One of the items in our system is regarded as "precious" in the sense that the total quantity of it produced by the system measures the payoff. The precious item could be skilled labor, completed assemblies, an input resource that is in scarce supply like a limited monetary budget. The contribution of each activity to the total payoff is the amount of the precious item that flows into or out of each activity. Thus, if the objective is to maximize profits, activities that require money contribute negatively and those that produce money contribute positively to total profits. The housewife's expenditures for each type of food, in § 1-2, Example 2, is a negative contribution to total "profits" of the household; there are no activities in this example that contribute positively. This characteristic of the linear programming model is known as the linear objective assumption.

The Standard Linear Programming Problem.

The determination of values for the *levels* of activities, which are positive or zero, such that flows of each item (for these activity levels) satisfy the material balance equations and such that the value of the payoff is a maximum is called the standard linear programming problem. The representation of a real system, as in any one of the three examples of § 1-2, as a mathematical system which exhibits the above characteristics, is called a linear programming model. The problem of programming the activities of the real system is thus transformed into the problem of finding the solution of the linear programming model.

3-2. BUILDING THE MODEL

Because model-building is an essential aspect of programming, the separate steps to be taken in building a linear programming model will now be systematized. We then show how the completed model defines the linear programming problem. The simplex method as a means for solving the general problem of linear programming will be dealt with in Chapter 5, but for the present we shall apply a less general method, the graphic, to two typical examples.

The mathematical model of a system is the collection of mathematical relationships which characterize the feasible programs of the system. By *feasible programs* is meant those programs which can be carried out under the system's limitations. Building a mathematical model often provides so much insight into a system and the organization of knowledge about a system that it is considered by many to be more important than the task of mathematical programming which it precedes. The model is often difficult to construct because of the richness, variety, and ambiguity of the real world. Nevertheless, it is possible to state certain principles which distinguish the separate steps in the model-building process.

The outline for this procedure presented below is based on the basic assumptions underlying the linear programming model of (a) proportionality, (b) nonnegativity, (c) additivity, and (d) a linear objective function. It is recommended that the reader review these concepts and identify these characteristics of the model in what follows.

Step 1: Define the Activity Set. Decompose the entire system under study into all of its elementary functions, the *activities*, and choose a unit for each activity in terms of which its quantity, or *level*, can be measured.

Step 2: Define the Item Set. Determine the classes of objects, the *items*, which are consumed or produced by the activities, and choose a unit for measuring each item. Select one item such that the net quantity of it produced by the system as a whole measures the "cost" (or such that its negative measures the "profit") of the entire system.[3]

[3] In the examples which follow, the "costs" happen to be money; however, in economic examples, they could be measured in terms of labor or any scarce resource, input which is to be conserved or any item whose total output from the system is to be maximized.

Step 3: Determine the Input-Output Coefficients. Determine the quantity of each item consumed or produced by the operation of each activity at its unit level. These numbers, the *input-output coefficients*, are the factors of proportionality between activity levels and item flows.

Step 4: Determine the Exogenous Flows. Determine the net inputs or outputs of the items between the system, taken as a whole, and the outside.

Step 5: Determine the Material Balance Equations. Assign unknown nonnegative activity levels $x_1, x_2, \ldots$, to all the activities; then, for each item, write the *material balance equation* which asserts that the algebraic sum of the flows of that item into each activity (given as the product of the activity level by the appropriate input-output coefficient) is equal to the exogenous flow of the item.

The result of the model-building is thus the collection of mathematical relationships characterizing all the feasible programs of the system. This collection is the *linear programming model.*

Once the model has been built, the linear programming problem can be posed in mathematical terms and its solution can be interpreted as a *program* for the system—a statement of the time and quantity of actions to be performed by the system so that it may move from its given status toward the defined objective.

The Linear Programming Problem.

Determine levels for all the activities of the system which (a) are nonnegative, (b) satisfy the material balance equations, and (c) minimize the total cost.

Devising techniques for solving the linear programming problem constitutes the central mathematical problem of linear programming, to which many of the succeeding chapters are devoted.

In our use of the steps for model-building in the examples below, one feature should be noted: namely, we will not always complete the model in one sequence of steps. It frequently happens that certain activities, commonly those related to the disposal of unused resources or the overfulfillment of requirements, are overlooked until the formulation of the material balance equations forces their inclusion. Thus a return from Step 5 to Step 1 will sometimes be necessary before the model is complete.

3-3. A TRANSPORTATION PROBLEM

In the cannery example of § 1-2 we required that the shipping schedule for cases minimize the total shipping cost from canneries to warehouses. To simplify that problem we shall suppose that there are two canneries, Cannery I and Cannery II, and three warehouses, labelled A, B, and C. The availability of cases at the canneries and the demands at the warehouses are as follows:

Cases Available	Cases Demanded
350 at Cannery I	300 at Warehouse A
650 at Cannery II	300 at Warehouse B
	300 at Warehouse C
1000 = Total available	900 = Total required

The excess production (100 cases) is to be stored without shipment. The shipping cost per case from either cannery to each warehouse is given in the Shipping Cost Schedule (1). The problem is to determine the number of cases each cannery should ship to each warehouse in order to minimize the total transportation cost.

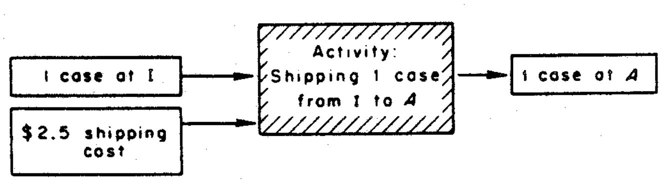
(1) Shipping Cost Schedule (dollars per case)

Canneries	Warehouses: New York (A)	Chicago (B)	Kansas City (C)
Seattle (I)	2.5	1.7	1.8
San Diego (II)	2.5	1.8	1.4

To formulate the model which describes the interrelations between the availabilities of cases at the canneries and demands at the warehouses, we shall begin by analyzing one of the elementary functions, namely the activity of *shipping from a cannery to a warehouse*. The activity of shipping a case from I to A (i.e., from Seattle to New York) is diagrammed in (2). It requires as *input* two items: one case in Seattle and \$2.5 expense. It produces as *output* one item: one case in New York. The basic assumption is that x cases to be shipped from I to A will require as inputs at I, $1 \cdot x$ cases, and $2.5x$ dollars in expenditures; it will produce as output $1 \cdot x$ cases at A.

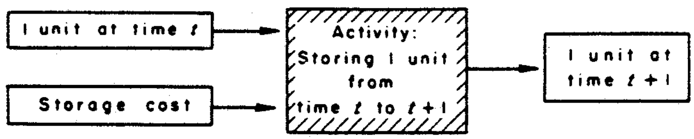
How this activity is performed, or what is done to a case between its origin and its destination, is not part of the programming problem. In this sense, then, the activity becomes a "black box" into which go certain items and out of which come other items; in this case, the output is a similar item, but at a different location.

(2) Black Box Diagram of a Transportation Activity



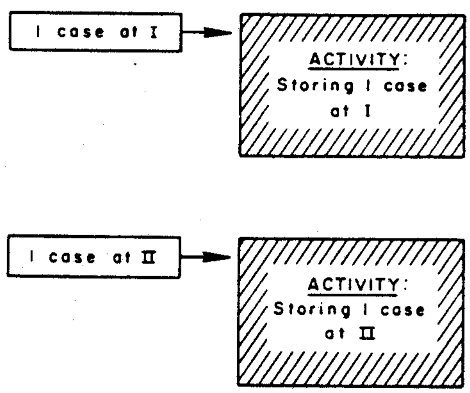
The cannery example contains six such shipping activities, which represent the six possible ways of shipping cases from two canneries to three warehouses. It is also possible to store production at the canneries, which leads to another kind of possible elementary function, the *storage activity*. A storage activity inputs an item and a cost (measured in dollars in this example, see § 3-2, footnote 3) at some time t and outputs the item at some later time $t + 1$.

(3) Black Box Diagram of a Typical Storage Activity



The similarity of the activities depicted in (2) and (3) occurs because the shipping activity is a transfer in *space*, while a storage activity is a transfer in *time*. Because in our particular problem we will not be considering the outputs at later times nor assigning any costs to storage, the two storage activities take on the simplified form (4).

(4)



Step 1: Let us now take the first step in formulating the model. We begin by listing in (5) the set of eight possible transportation and storage activities. For convenience the activities are assigned the reference numbers on the left; thus activity "4" is the activity of "Shipping from II to A." For the units to measure the quantity of either the shipping or storage activities, it is natural to choose *one case*; however, one could choose an entirely different kind of unit for each activity. For example, the unit of

the first activity could be tens of cases shipped and the second could be measured in dollars of transportation charges, etc.

(5)

Activity List

1. Shipping from I to A
2. ,, ,, I to B
3. ,, ,, I to C
4. ,, ,, II to A
5. ,, ,, II to B
6. ,, ,, II to C
7. Storing Excess at I
8. ,, ,, at II

Step 2: Except for costs it might be felt that only one other kind of item is available, namely a case. However, economists point out that similar items at different locations[4] or different times[5] are essentially different items. For our present purposes we are ignoring the time dimension and concentrating only on the different locations. Accordingly there will be a list of six items reflecting the two cannery locations, the three warehouse locations, and the cost item (money). The items shown in (6) are assigned the reference numbers on the left; thus item 4 is "Cases at B." The case will be used as the unit of measurement for each item 1–5, and the dollar will be used to measure costs, item 6.

(6)

Item List

1. Cases at I
2. ,, ,, II
3. ,, ,, A
4. ,, ,, B
5. ,, ,, C
6. Costs ($)

Step 3: In recording the input-output coefficients of flow for the model, this convention on the algebraic sign of the coefficient will be used: an input will be designated by a positive coefficient, and an output by a negative coefficient. Symbolically:

(7) $+ \longrightarrow$ [Activity] $\longrightarrow -$

We shall not, however, record the values of the coefficients in this form, but construct a coefficient table for them (see Table 3-3-I). There is one

[4] A bird in the hand is worth two in the bush.
[5] A stitch in time saves nine.

TABLE 3.3.1

COEFFICIENT TABLE—TRANSPORTATION MODEL

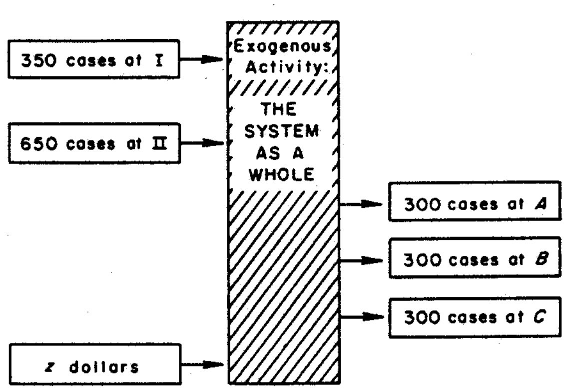
Items \ Activities	1 I → A	2 I → B	3 I → C	4 II → A	5 II → B	6 II → C	7 Store at I	8 Store at II
1. Cases at I	+1	+1	+1				+1	
2. Cases at II				+1	+1	+1		+1
3. Cases at A	−1			−1				
4. Cases at B		−1			−1			
5. Cases at C			−1			−1		
6. Costs ($)	+2.5	+1.7	+1.8	+2.5	+1.8	+1.4		

vertical *column* in this table for each activity, and one horizontal *row* for each item; at the intersection of each row and each column, we place the *signed* input-output coefficient for the flow of that item required by one unit of the activity.

Thus one unit of activity 4, shipping one case from II to A, has as inputs one case at II (coefficient +1 in row 2, column 4) and $2.5 (coefficient +2.5 in row 6, column 4); it has as output one case at A (coefficient −1 in row 3, column 4). This table is quickly checked by inspecting each row to see whether or not there has been a complete accounting of each item; thus in row 1, item 1 (cases at I) occurs only as an input, and that to activities 1, 2, 3, and 7; and in row 3, item 3 (cases at A) occurs only as output, of activities 1 and 4.

Step 4: Exogenous (outside) flows available to the system and required from the system as a whole are shown in (8) in "black box" form. The inputs

(8)



are the availabilities to the system at I and II and the outputs are the required flows from the system. Note that the dollar input has not yet been determined. *It is to be as small as possible.* Until it is determined, it will be denoted by "z."

It will be useful to write these exogenous flows in a column, ordered by item, similar to the column for each activity in Table 3-3-I. This is done in (9), where the same convention for the algebraic sign of exogenous flows must be used as for the flows into each activity within the system, because the algebraic sum of flow by activity will be equated to the exogenous flows. Therefore, exogenous inputs will be positive and exogenous outputs negative. Hence:

(9)

Item	Exogenous Flows	
1. Cases at I	350	available inputs into the system
2. Cases at II	650	
3. Cases at A	−300	required outputs from the system
4. Cases at B	−300	
5. Cases at C	−300	
6. Costs ($)	z	minimum input into the system

Step 5: With each activity 1, 2, . . ., 8 we associate an unknown quantity to be determined which represents the level of the activity. Customarily we denote the level of activity 1 by x_1, of activity 2 by x_2, . . ., of activity 8 by x_8.

Using the coefficient table generated in Step 3, it is now an easy matter to write the material balance equations for the system, item by item.

For item 1 (cases at I), the activities involved in its flow are 1, 2, 3, and 7 (shipping, storage at I). Because the input-output coefficients relating to item 1 are all +1, the net flow of item 1 is just

$$1 \cdot x_1 + 1 \cdot x_2 + 1 \cdot x_3 + 1 \cdot x_7$$

This flow must equal the exogenous flow of item 1 to the system, which is 350, yielding the first material balance equation,

$$x_1 + x_2 + x_3 + x_7 = 350$$

In precisely the same way, the material balance equation for item 2 (cases at II) is

$$x_4 + x_5 + x_6 + x_8 = 650$$

The equation has a different form for item 3 (cases at A). Here activities 1 and 4, which ship to A, have coefficients −1, and no other activities involve item 3. The net flow is

$$-1 \cdot x_1 - 1 \cdot x_4$$

and because the exogenous flow is the output −300, the equation is

$$-x_1 - x_4 = -300$$

The remaining equations, corresponding to items 4 and 5, give a similar accounting of cases at B and C respectively:

$$-x_2 - x_5 = -300$$
$$-x_3 - x_6 = -300$$

These equations are summarized in Table 3-3-II, Equations (11).

Finally, the flow of item 6 in the system is evidently given by

$$2.5x_1 + 1.7x_2 + 1.8x_3 + 2.5x_4 + 1.8x_5 + 1.4x_6$$

We place this in a material balance equation by setting it equal to an unspecified dollar input z. Recall that we do not yet know what numerical value z should have:

$$2.5x_1 + 1.7x_2 + 1.8x_3 + 2.5x_4 + 1.8x_5 + 1.4x_6 = z$$

Step 5 is now complete.

The Equation Form.

The set of material balance equations generated here, together with the conditions that all the activity levels $x_1, \ldots, x_8$ be nonnegative, constitutes the linear programming model for this transportation problem. These are summarized in (10) and (11) in what is referred to as the Equation Form of the model.

The Tableau.

The linear programming tableau affords both a compact form for writing the data of the linear programming model and a procedure for generating the material balance equations from these data without going through the detailed reasoning we have in Step 5.

The tableau for this problem is given in Table 3-3-II.

TABLE 3-3-II

LINEAR PROGRAMMING MODEL OF THE TRANSPORTATION PROBLEM

Tableau Form

Activities	I→A	I→B	I→C	II→A	II→B	II→C	Store at I	Store at II	Exogenous Flows
Items \ Levels	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	
1. Cases at I	1	1	1				1		350
2. Cases at II				1	1	1		1	650
3. Cases at A	−1			−1					−300
4. Cases at B		−1			−1				−300
5. Cases at C			−1			−1			−300
6. Costs ($)	2.5	1.7	1.8	2.5	1.8	1.4			z (Min)

Equation Form

(10) Nonnegativity $x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0,\ x_5 \geq 0,\ x_6 \geq 0,\ x_7 \geq 0,\ x_8 \geq 0$

(11) Material Balances

$$\begin{cases} x_1 + x_2 + x_3 + x_7 = 350 \\ x_4 + x_5 + x_6 + x_8 = 650 \\ -x_1 - x_4 = -300 \\ -x_2 - x_5 = -300 \\ -x_3 - x_6 = -300 \\ 2.5x_1 + 1.7x_2 + 1.8x_3 + 2.5x_4 + 1.8x_5 + 1.4x_6 = z\ (\text{Min}) \end{cases}$$

It consists of these parts:

(a) A list of the activities of the system and their unknown levels.
(b) A list of the items of the system.
The input-output coefficients of the system, arranged in columns by activity and in rows by item, as in the "Coefficient Table" of Table 3-3-I and later in the "Tableau Form" of Table 3-3-II.
(c) The exogenous flows to the system, in a column, as in (9).

The relationship in Table 3-3-II between the Equation Form of the model and the Tableau Form should be carefully noted. The tableau can be obtained from the equations by *detaching the coefficients* of the activity levels $x_1, \ldots, x_8$, that is, by suppressing the variables of the equations. When the model is presented in tableau form, the nonnegativity conditions (10) in Table 3-3-II will be understood to hold; on the other hand, the equations (11) can be immediately reconstructed from the tableau by forming, in each item-row, the products of the input-output coefficients with the appropriate unknown activity levels, summing across, and setting this expression for the net flow equal to the exogenous flow of the item.

The Linear Programming Problem.

Finally, we can state the mathematical problem for our particular example. Determine levels for the activities $x_1, x_2, \ldots, x_8$ which (a) are nonnegative (relations (10), Table 3-3-II), (b) satisfy the material balance equations (11), and (c) minimize z.

3-4. EXAMPLES OF BLENDING

A type of linear programming problem frequently encountered is one involving blending. Typically, different commodities are to be purchased, each having known characteristics and costs. The problem is to give a recipe showing how much of each commodity should be purchased and blended with the rest so that the characteristics of the mixture lie within specified bounds and the total purchase cost is minimized.

In the example we take up here, the characteristics of the blend are precisely specified. As will be seen later, only minor changes in the model are required in the event the blend specifications must lie between certain lower or upper bounds.

Blending Problem I.

A manufacturer wishes to produce an alloy which is 30 per cent lead, 30 per cent zinc, and 40 per cent tin. Suppose there are, on the market, alloys A, B, C, . . . with compositions and prices as given in (1). Per pound of blend produced, how much of each type of alloy should be purchased in order to minimize costs?

(1) Data for Blending Problem I

Alloy	A	B	C	D	E	F	G	H	I	Desired Blend
% Lead	10	10	40	60	30	30	30	50	20	30
% Zinc	10	30	50	30	30	40	20	40	30	30
% Tin	80	60	10	10	40	30	50	10	50	40
Costs/lb	$4.1	4.3	5.8	6.0	7.6	7.5	7.3	6.9	7.3	Min

Obviously the manufacturer can purchase alloy E alone, but it costs $7.60 per pound. If he buys $\frac{1}{4}$ pound each of alloys A, B, C, and D, he gets one pound of a 30-30-40 mixture at a cost of $5.05; $\frac{1}{4}$ pound of A, $\frac{1}{4}$ pound of B, and $\frac{1}{2}$ pound of H again give one pound of mixture with correct proportions, but costs $5.55. After a few trials of this sort, the manufacturer may well seek a more general approach to his problem.

In formulating the linear programming model for this example, we must first note that the blending problem has not been posed as completely as, say, the transportation problem of the preceding section. The *quantities* of lead, zinc, and tin in the final blend have not been specified, only their proportions have been given, and it is required to minimize the cost per pound of the output. Because we need specific data for the exogenous flows, we shall require that a definite amount of blended metal be produced. It is clear that a recipe giving the most economical purchasing program for one pound of blended metal output can be immediately converted into a recipe giving the most economical purchasing program for n pounds of output by multiplying the levels of all the activities involved by n; and thus we will restrict the *quantity* of *activities to those combinations which produce one pound of blended metal.* This restriction is expressed later, implicitly in the statement of exogenous flows (6), and again explicitly in the material balance equations (8).

This stipulation has the further happy result that the percentage requirements of the original statement of the problem now become concrete: the mixture must contain 0.3 pounds of lead, 0.3 pounds of zinc, and 0.4 pounds of tin. (Often a beginner attempts to formulate the problem without restricting the total amount produced, in which case the material balance equations become difficult to interpret, being expressed in terms of percentages instead of amounts.)

Step 1: Identifying activities. The only activities we need to consider are those of purchasing each of the nine alloys, because we assume all the metal purchased will be blended. The unit level for each activity will be the purchase of one pound of the alloy.

(2) Activity List

1. Purchasing alloy A; activity level x_1
2. ,, ,, B ,, ,, x_2
3. ,, ,, C ,, ,, x_3
4. ,, ,, D ,, ,, x_4
5. ,, ,, E ,, ,, x_5
6. ,, ,, F ,, ,, x_6
7. ,, ,, G ,, ,, x_7
8. ,, ,, H ,, ,, x_8
9. ,, ,, I ,, ,, x_9

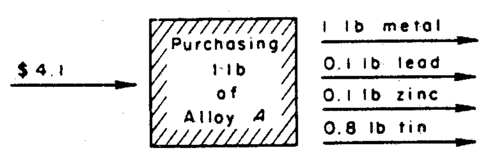
Step 2: Identifying items. The items considered in the system can now be listed:

(3) Item List

1. Metal (*total*) measured in pounds
2. Lead ,, ,, ,,
3. Zinc ,, ,, ,,
4. Tin ,, ,, ,,
5. Cost ,, ,, dollars

Step 3: Input-output coefficients. We shall adopt the first of the three points of view discussed in the footnote[6] in what follows. A typical activity —say activity 1, purchasing alloy A—has the appearance

(4)



using the data of (1). Each of the nine activities has likewise one input and four outputs. Each activity has, of course, one pound of metal as one

[6] There are three points of view that one can take in formulating this model: (1) the viewpoint of the alloy purchaser is that he receives dollars and outputs contributions to pounds of finished blend and to the lead, tin, zinc characteristics; (2) the viewpoint of the blender is that he inputs contributions to lead, tin, zinc characteristics and outputs dollars and pounds of finished blend; (3) the viewpoint of the receiver of the finished blend is that he receives finished metal and contributions to lead, tin, zinc characteristics and outputs money.

output; the remaining entries in Table 3-4-I, of input-output coefficients are extracted directly from the data in (1).

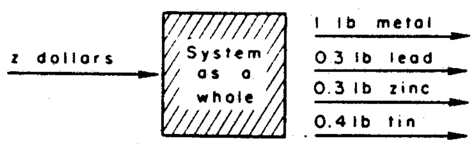
TABLE 3-4-I

COEFFICIENT TABLE: BLENDING PROBLEM I

Items \ Activities	1 A	2 B	3 C	4 D	5 E	6 F	7 G	8 H	9 I
1. Metal	−1	−1	−1	−1	−1	−1	−1	−1	−1
2. Lead	−0.1	−0.1	−0.4	−0.6	−0.3	−0.3	−0.3	−0.5	−0.2
3. Zinc	−0.1	−0.3	−0.5	−0.3	−0.3	−0.4	−0.2	−0.4	−0.3
4. Tin	−0.8	−0.6	−0.1	−0.1	−0.4	−0.3	−0.5	−0.1	−0.5
5. Costs ($)	4.1	4.3	5.8	6.0	7.6	7.5	7.3	6.9	7.3

Step 4: Exogenous flows. These are shown in "black box" form in (5), and as a list in (6):

(5) Exogenous Flows—Blending Problem I



(6)

Item	Exogenous Flows
1. Metal	−1.0
2. Lead	−0.3
3. Zinc	−0.3
4. Tin	−0.4
5. Costs ($)	z (Min)

Step 5: Material balance equations. As noted in § 3-3, the Equation Form for the model can be assembled directly from the results of Steps 3 and 4. Combining the coefficient table (Table 3-4-I) and the exogenous flow list (6), we arrive at the Tableau Form of our model shown in Table 3-4-II.

Linear Programming Problem for Blending Model I. Determine levels for the activities $x_1, x_2, \ldots, x_9$ which (a) are nonnegative (relations (7), Table 3-4-II), (b) satisfy the material balance equations (8), and (c) minimize z.

TABLE 3-4-II

LINEAR PROGRAMMING MODEL OF BLENDING PROBLEM I

Tableau Form

Activities	A	B	C	D	E	F	G	H	I	Exogenous Flows
Items \ Buy at level	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	
1. Metal (total)	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
2. Lead	− .1	− .1	− .4	− .6	− .3	− .3	− .3	− .5	− .2	− .3
3. Zinc	− .1	− .3	− .5	− .3	− .3	− .4	− .2	− .4	− .3	− .3
4. Tin	− .8	− .6	− .1	− .1	− .4	− .3	− .5	− .1	− .5	− .4
5. Costs ($)	4.1	4.3	5.8	6.0	7.6	7.5	7.3	6.9	7.3	z (Min)

Equation Form

(7) Non-negativity $\quad x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0,\ x_5 \geq 0,\ x_6 \geq 0,\ x_7 \geq 0,\ x_8 \geq 0,\ x_9 \geq 0$

(8) Material Balances

$$\begin{cases} -x_1 - x_2 - x_3 - x_4 - x_5 - x_6 - x_7 - x_8 - x_9 = -1 \\ -.1x_1 - .1x_2 - .4x_3 - .6x_4 - .3x_5 - .3x_6 - .3x_7 - .5x_8 - .2x_9 = -.3 \\ -.1x_1 - .3x_2 - .5x_3 - .3x_4 - .3x_5 - .4x_6 - .2x_7 - .4x_8 - .3x_9 = -.3 \\ -.8x_1 - .6x_2 - .1x_3 - .1x_4 - .4x_5 - .3x_6 - .5x_7 - .1x_8 - .5x_9 = -.4 \\ 4.1x_1 + 4.3x_2 + 5.8x_3 + 6.0x_4 + 7.6x_5 + 7.5x_6 + 7.3x_7 + 6.9x_8 + 7.3x_9 = z\ (\text{Min}) \end{cases}$$

Blending Problem II.

The particular linear programming problem considered above is a little too large for us to solve conveniently until the techniques of Chapter 5 have been developed. (It is given as the Illustrative Example 2 of that chapter.) Its solution is found to be $x_1 = 0$, $x_2 = \frac{3}{5}$, $x_4 = \frac{2}{5}$, and all the remaining activities at zero level. The minimum cost for one pound of metal is $4.98. As an alternative we shall consider an easier and different problem.

To simplify the blending problem so that it can be solved here graphically, let us try to find the cheapest blend of alloys that will have .4 lb. of tin per pound of metal (the remaining .6 lb. of metal may have lead and zinc in any ratio). This is, of course, not the problem we formulated earlier, but it will not be necessary to go through the whole model-building process again in order to formulate it. All we have done is to drop here the requirements laid down in (6) for items 2 (lead) and 3 (zinc); the other requirements, the activities and the input-output coefficients, need not be changed in building this simpler model. Thus, we can obtain the equation form of the simplified model by merely deleting the second and third equations of (8), which relate to lead and zinc. We are left with the first, fourth, and fifth equations of (8).

The discussion will be made still easier if we change the signs of all the terms in the "Metal" and "Tin" equations.

Linear Programming Problem for Blending Model II. Determine levels for the activities $x_1, x_2, \ldots, x_9$ which (a) are nonnegative, (b) satisfy the equations

$$\begin{array}{llllllllllll}
(9) & x_1 + & x_2 + & x_3 + & x_4 + & x_5 + & x_6 + & x_7 + & x_8 + & x_9 & = & 1 \\
(10) & .8x_1 + & .6x_2 + & .1x_3 + & .1x_4 + & .4x_5 + & .3x_6 + & .5x_7 + & .1x_8 + & .5x_9 & = & .4 \\
(11) & 4.1x_1 + & 4.3x_2 + & 5.8x_3 + & 6.0x_4 + & 7.6x_5 + & 7.5x_6 + & 7.3x_7 + & 6.9x_8 + & 7.3x_9 & = & z
\end{array}$$

and (c) minimize z.

Graphical Representation. The data of the blending problem have now been reduced sufficiently to permit their graphical representation in Fig. 3-4-I. For each of the nine activities we take its two coefficients from equations (10) and (11), and represent the activity by a point having these two numbers as coordinates. Thus the point A, representing alloy A, has coordinates (.8, 4.1), which are the amount of tin and the cost per pound of alloy A; similarly, the point B has coordinates (.6, 4.3), the amount of tin and cost per pound of alloy B; etc. Let (u, v) be the coordinates of a general point.

The fact which makes this graphical representation valuable is that not only can the input-output coefficients of any activity be represented by a point, but the net exogenous flow to the system as a whole can be represented also as a point for any program involving nonnegative levels $x_1, \ldots, x_9$, which *sum to unity*. Consider, for example, the program $x_1 = x_2 = \frac{1}{2}$, $x_3 = x_4 = \ldots = 0$, which consists of using one-half pound each of alloys A and B. It yields $.8(\frac{1}{2}) + .6(\frac{1}{2}) = 0.7$ pound of tin and costs $4.1(\frac{1}{2}) + 4.3(\frac{1}{2}) = 4.2$, and can thus be represented in Fig. 3-4-I by the point p_1, half-way between A and B. Another program, $x_1 = \frac{1}{2}$, $x_2 = x_9 = \frac{1}{4}$, using one-half pound of A and one-quarter each of B and I, has coordinates $.8(\frac{1}{2}) + .6(\frac{1}{4}) + .5(\frac{1}{4}) = 0.675$ for tin and $4.1(\frac{1}{2}) + 4.3(\frac{1}{4}) + 7.3(\frac{1}{4}) = 4.95$ for cost, and can be represented by p_2.

In each case, the coordinates of the point representing the mixture are a *weighted average* of the corresponding coordinates of the points representing the pure alloys; thus, we say that the point p_1 is the weighted average of the points A and B with weights $\frac{1}{2}$ and $\frac{1}{2}$, respectively, and that p_2 is the weighted average of the points A, B, and I with weights $\frac{1}{2}$, $\frac{1}{4}$, and $\frac{1}{4}$, respectively. (In physics, p_1 is said to be the *center of gravity* of the system consisting of a weight of $\frac{1}{2}$ unit at A and $\frac{1}{2}$ unit at B; likewise, p_2 is the center of gravity of the system consisting of weights $\frac{1}{2}$, $\frac{1}{4}$, and $\frac{1}{4}$ at A, B, and I, respectively.)

It should now be clear that all the nonnegative programs satisfying just relation (9) are represented by the shaded region of Fig. 3-4-I, the collection of all possible weighted averages of the nine points A, . . ., I. The *feasible programs*, however, are those which yield exactly 0.4 pound of tin; they are represented by the points of the shaded region which lie on the vertical line having abscissa 0.4. The point E is such a program, as well as the point $R = (0.4, 5.55)$, which is the weighted average of A, B, and H with weights $\frac{1}{4}$, $\frac{1}{4}$, and $\frac{1}{2}$, respectively. Evidently neither of these points represents the least-z solution of the problem; the point which does, is the lowest point on the vertical line which is in the shaded region. Thus, the linear programming problem can be interpreted graphically as one of assigning nonnegative weights to the vertices of the figure in such a way that the weighted average of the vertices lies on the vertical line whose abscissa is 0.4 and has as

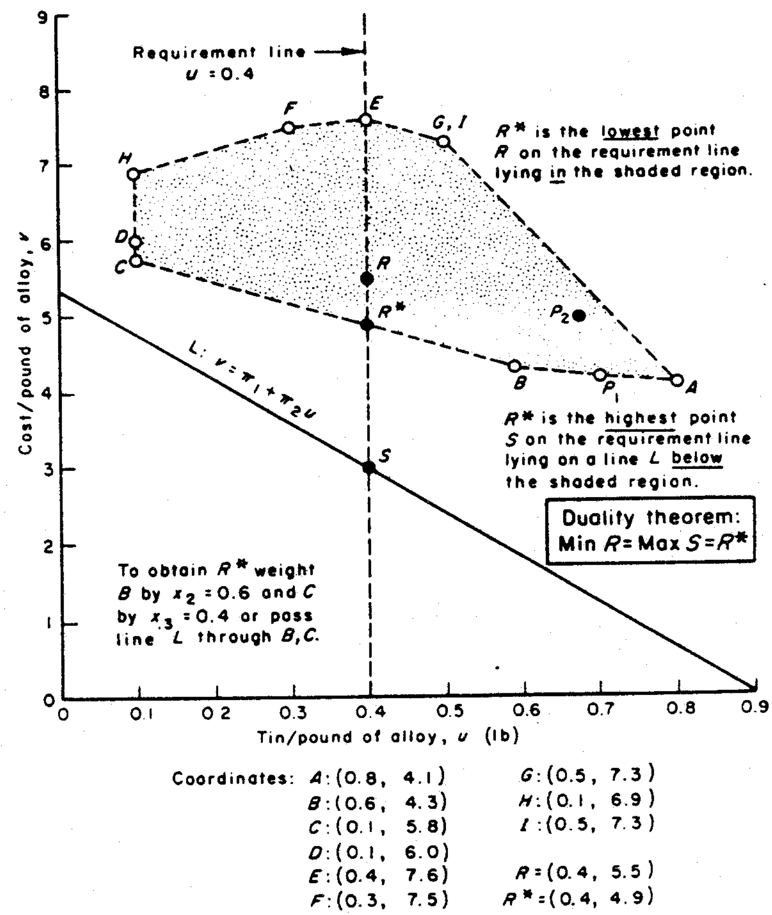


Figure 3-4-I. Duality theorem illustrated on blending model II.

small an ordinate z as possible. From the graph we can see that the desired weighted average, R*, lies on the line BC; i.e., it is the average obtained by giving certain weights, x_2 and x_3, to B and C, and weights of zero to all the others. To determine x_2 and x_3, set all $x_j = 0$ except x_2 and x_3 in (9) and (10), i.e., consider mixtures consisting only of B and C; then we must have

$$x_2 + x_3 = 1$$
$$.6x_2 + .1x_3 = .4$$

which yields

$$x_2 = .6,\ x_3 = .4$$

and

$$\text{Min } z = 4.3x_2 + 5.8x_3 = 4.9$$

We conclude that it is best to blend in the proportions of .6 pound of alloy B to .4 pound of alloy C to produce the cheapest alloy containing 40 per cent tin. The blend will cost \$4.9 per pound.

Algebraic Check—the Dual Linear Program. We can check algebraically whether our choice of B, C in Fig. 3-4-I is correct by first determining the line joining B to C and then testing to see if each of the points of the shaded region has an ordinate value v greater than that of the point on the line with the same abscissa u. If the latter is true we say the shaded region lies "above" the extended line joining B to C.

Now the equation of a *general line* in the (u, v) plane is

$$v = \pi_1 + \pi_2 u$$

where π_1 is the *intercept* and π_2 the *slope*. In order that the shaded region lie above this line, each of the points A, B, C, . . ., G, I (which generated the shaded region) must lie on or above the line. Substituting the $u = .8$ coordinate of A into the equation, the value $v = \pi_1 + \pi_2(.8)$ must be *less than or equal to* the v coordinate of A. Thus our test for A is $\pi_1 + \pi_2(.8) \leq 4.1$ and for the entire set A, B, C, . . . we must have

$$\begin{aligned}
(12) \qquad \pi_1 + \pi_2(.8) &\leq 4.1\\
\pi_1 + \pi_2(.6) &\leq 4.3\\
\pi_1 + \pi_2(.1) &\leq 5.8\\
\pi_1 + \pi_2(.1) &\leq 6.0\\
\pi_1 + \pi_2(.4) &\leq 7.6\\
\pi_1 + \pi_2(.3) &\leq 7.5\\
\pi_1 + \pi_2(.5) &\leq 7.3\\
\pi_1 + \pi_2(.1) &\leq 6.9\\
\pi_1 + \pi_2(.5) &\leq 7.3
\end{aligned}$$

Let $S = (.4,\ \bar{v})$ be the intersection of the vertical line $u = .4$ with $v = \pi_1 + \pi_2 u$; then the line we are looking for (and which we hope will be the one joining B to C) is the one below the shaded region whose $v = \bar{v}$ coordinate of S is *maximum*, i.e.,

$$(13) \qquad \pi_1 + \pi_2(.4) = \bar{v}\ (\text{Max})$$

The problem of finding π_1, π_2 and Max $\bar{v}$ satisfying (12) and (13) is known as the *dual* of our original (*primal*) problem (9), (10), and (11). The fact that Max $\bar{v}$ = Min z for these two problems is a particular case of the Duality Theorem for Linear Programs (see Chapter 6).

If we conjecture that some pair like B, C (obtained by visual inspection of the graph or otherwise) is an optimal choice, it is an easy matter to verify this choice by checking whether (i) the intersection S lies *between* the selected two points and (ii) all points A, B, C, . . . lie *on or above* the extended line joining the selected two points. To check the first, we solve

(14)
$$x_2 + x_3 = 1$$
$$.6x_2 + .1x_3 = .4$$

obtaining $x_2 = .6$, $x_3 = .4$ which are positive, so that S lies between B and C. Thus these values with remaining $x_i = 0$ satisfy the primal system (9), (10), and (11). To check the second we determine the equation of the line by stating the conditions that the line pass through B and C,

(15)
$$\pi_1 + \pi_2(.6) = 4.3$$
$$\pi_1 + \pi_2(.1) = 5.8$$

This yields $\pi_1 = 6.1$, $\pi_2 = -3$, which satisfy the dual system (12).

3-5. A PRODUCT MIX PROBLEM

A furniture company manufactures four models of desks. Each desk is first constructed in the carpentry shop and is next sent to the finishing shop, where it is varnished, waxed, and polished. The number of man hours of labor required in each shop is as follows:

(1)

	Desk 1	Desk 2	Desk 3	Desk 4
Carpentry Shop	4	9	7	10
Finishing Shop	1	1	3	40

Because of limitations in capacity of the plant, no more than 6,000 man hours can be expected in the carpentry shop and 4,000 in the finishing shop in the next six months.

The profit (revenue minus labor costs) from the sale of each item is as follows:

(2)

Desk	1	2	3	4
Profit	$12	$20	$18	$40

Assuming that raw materials and supplies are available in adequate supply and all desks produced can be sold, the desk company wants to determine the optimal product mix, i.e., the quantities to make of each type product which will maximize profit.

Step 1: Activities. The four manufacturing activities are

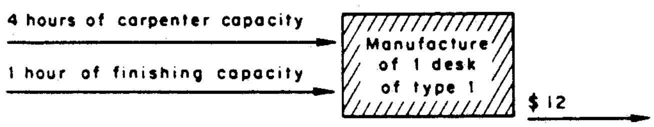
1. Manufacturing desk 1 (measured in desks produced)
2. ,, ,, 2 (,, ,, ,, ,,)
3. ,, ,, 3 (,, ,, ,, ,,)
4. ,, ,, 4 (,, ,, ,, ,,)

Step 2: Items.

1. Capacity in Carpentry Shop (measured in man hours)
2. Capacity in Finishing Shop (measured in man hours)
3. Costs (measured in dollars)

Step 3: Coefficients. Manufacturing activity 1, for example, can be diagrammed as follows:

(3)



The table of input-output coefficients constructed from (1) and (2) is shown in Table 3-5-I.

TABLE 3-5-I

COEFFICIENT TABLE: PRODUCT MIX PROBLEM

Activities / Items	Manufacturing Desks			
	(1)	(2)	(3)	(4)
1. Carpentry capacity (hours)	4	9	7	10
2. Finishing capacity (hours)	1	1	3	40
3. Cost ($)	−12	−20	−18	−40

Step 4: Exogenous flows. Since capacities, in carpentry and finishing, are inputs to each of these activities, they must be inputs to the system as a whole. At this point, however, we must face the fact that a feasible program need not use up all of this capacity. The total inputs must not be more than 6,000 carpentry hours and 4,000 finishing hours, but they can be less, and so cannot be specified precisely in material balance equations.

Step 5: Material balances. If we went ahead with the formulation anyway, using these figures for the exogenous flows, then in order to retain reality in the mathematical formulation, we should have to write material balance inequalities instead of equations, expressing, for example, the carpentry capacity limitation as

$$4x_1 + 9x_2 + 7x_3 + 10x_4 \leq 6000$$

instead of as an equation, which is not according to our rules.

We see that the model cannot be completed with the lists of activities and items given above, and we have here the case mentioned in the first section in which a second pass at the initial building of the model is necessary.

In this instance all we need to do is add activities to the model which will account for the carpentry and finishing capacity not used by the remainder of the program. If we specify "not using capacity" as an activity, we have the two additional activities to add to those listed in Step 1:

5. Not using Carpentry Shop capacity (measured in man hours)
6. Not using Finishing Shop capacity (measured in man hours)

Activity 5 can be abstracted as

(4)

The full tableau of inputs and outputs of the activities and the exogenous availabilities to the system as a whole is shown in Table 3-5-II.

TABLE 3-5-II

LINEAR PROGRAMMING PROBLEM FOR A PRODUCT MIX MODEL

Activities / Items	Manufacturing Desks				Not Using Capacity Carp.	Not Using Capacity Fin.	Exogenous Flows Input (+) Output (−)
	(1) x_1	(2) x_2	(3) x_3	(4) x_4	(5) x_5	(6) x_6	
1. Carpentry capacity (hours)	4	9	7	10	1		6000
2. Finishing capacity (hours)	1	1	3	40		1	4000
3. Costs ($)	−12	−20	−18	−40			z (Min)

Thus the programming problem is to determine numbers

(5) $$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0,\ x_5 \geq 0,\ x_6 \geq 0$$

and minimum z satisfying

$$(6)\qquad \begin{aligned} 4x_1 + 9x_2 + 7x_3 + 10x_4 + x_5 &= 6000,\\ x_1 + x_2 + 3x_3 + 40x_4 + x_6 &= 4000,\\ -12x_1 - 20x_2 - 18x_3 - 40x_4 &= z \end{aligned}$$

Note that the same values of the x's which minimize the cost function will also maximize its negative, namely the profit function p given by

$$+12x_1 + 20x_2 + 18x_3 + 40x_4 = p$$

Thus, a profit maximization problem can be stated as an equivalent to a cost minimization problem.

Graphical Solution. To apply the method of solution of the last section to the product mix model, it is necessary to change the definitions of items and activity levels so that the activity levels sum to unity. This is simply done by introducing as an item, *total capacity*, which is the sum of the carpentry capacity and the finishing capacity, and *changing units* for measuring activity levels so that 1 new unit of each activity requires the full 6000 + 4000 = 10,000 hours of total capacity. To change units note that one unit of the first activity in Table 3-5-II requires 5 hours of total capacity; thus, 2,000 units of the first activity would require 10,000 hours of capacity and is equivalent to *one new unit* of the first activity. In general, if y_1 is the number of new units of the first activity, then $2000y_1 = x_1$. The relationships between the old and new activity levels after such a change in units for each activity is

$$2000y_1 = x_1,\quad 1000y_2 = x_2,\quad 1000y_3 = x_3,$$
$$200y_4 = x_4,\ 10{,}000y_5 = x_5,\ 10{,}000y_6 = x_6.$$

It is also convenient to change the units for measuring capacity and costs. Let 10,000 hours = 1 new capacity unit; \$10,000 = 1 new cost unit. Then it is easy to see (and this is left as an exercise) that the Product Mix Model Table 3-5-II will become Table 3-5-III after the changes in the units for

TABLE 3-5-III

A PRODUCT MIX MODEL (after change in units)

Activities / Items	Manufacturing Desks (1 = 10,000 hours)				Not Using Capacity		Exogenous Flows
					Carp.	*Fin.*	
	(1) y_1	(2) y_2	(3) y_3	(4) y_4	(5) y_5	(6) y_6	
0. Total capacity (1 = 10,000 hours)	1.0	1.0	1.0	1.0	1.0	1.0	1.0
1. Carpentry capacity	.8	.9	.7	.2	1.0		.6
2. Finishing capacity	.2	.1	.3	.8		1.0	.4
3. Costs (1 = \$10,000)	−2.4	−2.0	−1.8	− .8			z' (Min)

activities and items given above, the replacing of z by 10,000 [illegible] in the cost equation, and the adding of the two equations to form a total capacity equation.

We are now ready to find the graphical solution. Because the unknowns $y_j \geq 0$ sum to unity, we shall interpret this as assigning nonnegative weights to points $A_1, A_2, \ldots, A_6$ in Fig. 3-5-I. As in the blending problem of the

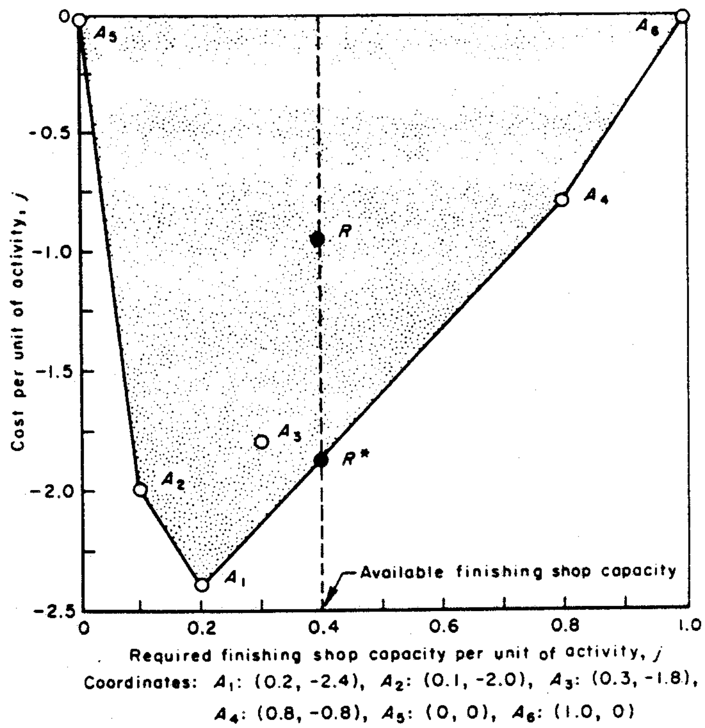


Figure 3-5-I. Graphical solution of the product mix problem.

preceding section, we shall ignore one of the material balance equations, namely that for item 1, carpentry capacity; however, here we will find that ignoring it does not affect the minimal solution because the equation is redundant.

In Fig. 3-5-I each point A_j corresponds to a column, or activity, of Table 3-5-III; its coordinates are the coefficients for the *finishing* capacity and cost of the activity. Thus the coordinates of A_1 are (.2, −2.4); of A_2 are (.1, −2.0), . . .; of A_5 are (0, 0); and of A_6 are (1.0, 0).

We seek an assignment of nonnegative weights y_j for each of the six points which sum to unity, so that their weighted average has coordinates (.4, z') and z' is minimal. This, clearly, is the point R* found by assigning zero weights to all points, except A_1 and A_4, and appropriately weighting the latter so that the center of gravity of A_1 and A_4 has abscissa 0.4. To determine y_1 and y_4, set all $y_j = 0$ except y_1 and y_4 in Table 3-5-III, yielding

$$.2y_1 + .8y_4 = .4$$
$$y_1 + y_4 = 1$$
$$-2.4y_1 - .8y_4 = z'$$

whence

$$y_1 = 2/3,\ y_4 = 1/3,\ z' = -5.6/3$$

Thus the optimal solution is to manufacture $x_1 = \frac{2}{3}(2000)$ desks of Type 1, $x_4 = \frac{1}{3}(200)$ desks of Type 4, which will use the full capacity of the plant and will cost $z = \$10,000\ (-5.6/3)$, or yield \$18,666.66 profit.

The carpentry capacity is completely accounted for by this solution, despite the fact that its material balance equation was omitted in the above calculation. As noted earlier, this is because adding the *total capacity* equation to the system enables us to drop either of the remaining equations and still have a model which accounts for all the capacities; the carpentry capacity equation becomes redundant, and can be dropped.

3-6. A SIMPLE WAREHOUSE PROBLEM

Consider the problem of stocking a warehouse with a commodity for sale at a later date. The warehouse can stock only 100 units of the commodity. The storage costs are \$1.00 per quarter for each unit. In each quarter the purchase price equals the selling price. This price varies from quarter to quarter according to (1):

(1)

Quarter (t)	Price per unit (dollars)
1	10
2	12
3	8
4	9

This implies that a profit can be realized by buying when the price is low and selling when the price is high. The problem is to determine the optimal selling, storing, and buying program for a one-year period by quarters, assuming that the warehouse has an initial stock of 50 units.

In each period (quarter), t, we distinguish four types of activities:

	Quantity
1. Selling stock	x_{t1}
2. Storing stock	x_{t2}
3. Buying stock	x_{t3}
4. Not using capacity (slack)	x_{t4}

and three types of items:

1. Stock
2. Storage Capacity
3. Costs

These activities have the input-output characteristics sketched in (2).

With four time periods each item and activity is repeated four times, which leads to Table 3-6-I, the tableau for the warehouse problem. The problem here is to find the values of $x_{ti} \geq 0$ which satisfy the equations implied by the tableau and which minimize the total cost.

(2)

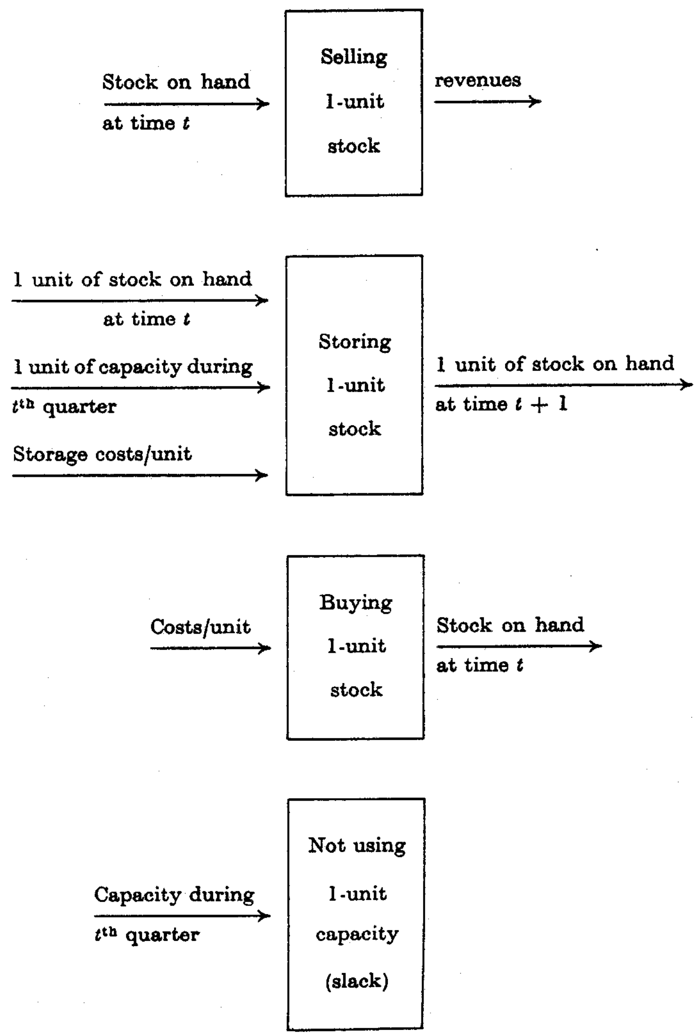


TABLE 3-6-I

A SIMPLE WAREHOUSE MODEL

Activities / Items	1st Quarter Sell x_{11}	Store x_{12}	Buy x_{13}	Slack x_{14}	2nd Quarter Sell x_{21}	Store x_{22}	Buy x_{23}	Slack x_{24}	3rd Quarter Sell x_{31}	Store x_{32}	Buy x_{33}	Slack x_{34}	4th Quarter Sell x_{41}	Store x_{42}	Buy x_{43}	Slack x_{44}	Exogenous Flows
$t = 0$ Stock	1	1	−1														50
Capac.		1		1													100
$t = 1$ Stock		−1			1	1	−1										0
Capac.						1		1									100
$t = 2$ Stock						−1			1	1	−1						0
Capac.										1		1					100
$t = 3$ Stock										−1			1	1	−1		0
Capac.														1		1	100
Costs	−10	1	10		−12	1	12		−8	1	8		−9	1	9		z (Min)

3-7. ON-THE-JOB TRAINING

The purpose of this example is to illustrate the ability of the linear programming model to cover the many and varied conditions that are so characteristic of practical applications.

The problem. A manufacturing plant has a contract to produce 1200 units of some commodity, C, with the required delivery schedule r_t as in (1).

(1)

End of week	1	2	3	4	5
No. of units	$r_1 = 100$	$r_2 = 200$	$r_3 = 300$	$r_4 = 400$	$r_5 = 200$

What hiring, firing, producing, and storing schedule should the manufacturer adopt to minimize the costs of his contract under the following conditions?

(a) Each unit of production not delivered on schedule involves a penalty of $p = \$30$ per week until delivery is effected.

(b) Any production ahead of schedule requires storage at $s = \$10$/unit/week.

(c) All required deliveries must be met by the end of the fifth week.

(d) Initially there are $g = 20$ workers and $h = 10$ units of C on hand.

(e) Each worker used in production during a week can turn out $k = 8$ units of C.

(f) Each worker used for training recruits during a week can train $\ell - 1 = 5$ new workers (i.e., produce ℓ trained workers including himself).

(g) Wages of a worker are $m = \$100$/week when used in production or when idle.

(h) Wages of a worker plus $\ell - 1$ recruits used in training for one week are $n = \$600$.

(i) The cost to fire one worker is $f = \$100$.

We shall choose for our unit of time a period of one week. At the beginning of each week we shall assign the necessary number of workers and units of C to carry out an activity that takes place during the week. Accordingly, at each of the six times $t = 0, 1, 2, \ldots, 5$, material balance equations for two items will be set up:

	Symbol for item
Workers	W_t
Commodity	C_t

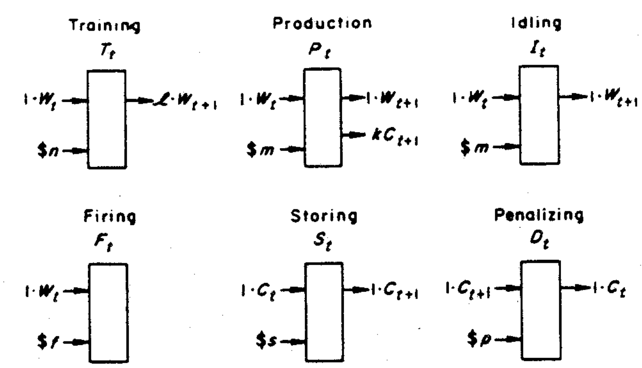
In addition to these equations there will be a cost equation for the *cost item*. In each of five weekly periods six activities will be set up as in (2).

(2)

	Symbol for Activity
1. Training	T_t
2. Producing	P_t
3. Idling	I_t
4. Firing	F_t
5. Storing	S_t
6. Penalizing (for Deficit)	D_t

The input-output characteristics of each of the activities, except perhaps the penalizing activity, are straightforward. Each failure to deliver a unit makes it necessary to decrease by one unit the present demand for the commodity and to increase the demand one unit in the next time period at a cost of p dollars. Another rationalization of this activity is to imagine that the deficit is temporarily satisfied by renting on the open market one unit of the commodity which must be returned the following week at a cost of p dollars.

(3)



These activities are shown in conventional tableau form in Table 3-7-I. In the fifth week the penalizing activity is omitted because condition (c) states that all deliveries must be met by the end of the fifth week. In the sixth week a firing activity F_6 has been introduced to get rid of all workers and to terminate the program. (Why is this necessary?)

TABLE 3-7-I

THE JOB TRAINING MODEL

Item	1st Week						2nd Week						3rd Week						4th Week						5th Week						Exogenous Flows
	T_1 x_{11}	P_1 x_{12}	I_1 x_{13}	F_1 x_{14}	S_1 x_{15}	D_1 x_{16}	T_2 x_{21}	P_2 x_{22}	I_2 x_{23}	F_2 x_{24}	S_2 x_{25}	D_2 x_{26}	T_3 x_{31}	P_3 x_{32}	I_3 x_{33}	F_3 x_{34}	S_3 x_{35}	D_3 x_{36}	T_4 x_{41}	P_4 x_{42}	I_4 x_{43}	F_4 x_{44}	S_4 x_{45}	D_4 x_{46}	T_5 x_{51}	P_5 x_{52}	I_5 x_{53}	F_5 x_{54}	S_5 x_{55}	F_6 x_{64}	
W_0	1	1	1	1																											g
C_0					1	−1																									h
W_1	$-\ell$	−1	−1				1	1	1	1																					0
C_1		$-k$			−1	1					1	−1																			$-r_1$
W_2							$-\ell$	−1	−1				1	1	1	1															0
C_2								$-k$			−1	1					1	−1													$-r_2$
W_3													$-\ell$	−1	−1				1	1	1	1									0
C_3														$-k$			−1	1					1	−1							$-r_3$
W_4																			$-\ell$	−1	−1				1	1	1	1			0
C_4																				$-k$			−1	1					1		$-r_4$
W_5																									$-\ell$	−1	−1			1	0
C_5																										$-k$			−1		$-r_5$
Cost	n	m	m	f	s	p	n	m	m	f	s	p	n	m	m	f	s	p	n	m	m	f	s	p	n	m	m	f	s	f	z (Min)

3-8. THE CENTRAL MATHEMATICAL PROBLEM

In the preceding sections, linear programming models were constructed for several examples. In each of these the problem was to find the solution of a system of linear equations or inequalities which minimized or maximized a linear form. This optimizing of a linear form, subject to linear restraints, is called the central mathematical problem of linear programming.

Whenever the restraints were stated as inequalities in the examples, it was possible to change each inequality to an equation by the addition of a *slack variable*. Furthermore, a problem in which a linear function was to be maximized could be converted to a problem of minimizing the negative of this form.

Thus, it is possible to formulate all linear programming problems in the same general manner; namely, to find the solution of a system of linear equations in nonnegative variables which minimizes a linear form. Since this algebraic statement of the problem arises naturally in many applications, it is called the "standard form" of the linear programming problem. In this section the formulation of the standard form of the central mathematical problem of linear programming will be reviewed and formalized.

If the subscript $j = 1, 2, \ldots, N$ denotes the j^{th} type of activity and x_j its quantity (or activity level), then usually $x_j \geq 0$. If, for example, x_j represents the quantity of a stockpile allocated for the j^{th} use, it does not, as a rule, make sense to allocate a negative quantity. In certain cases, however, one may wish to interpret a negative quantity as meaning taking stock from the j^{th} use. Here some care must be exercised; for example, there may be costs, such as transportation charges, which are positive regardless of the direction of flow of the stock. One must also be careful not to overdraw the stock of the using activity. For these reasons it is better in formulating models to distinguish two activities, each with a nonnegative range, for their respective x_j, rather than to try incorporating them into a single range.

The interdependencies between various activities arise because all practical programming problems are circumscribed by commodity limitations of one kind or another. The limited commodity may be raw materials, manpower, facilities, or funds; these are referred to by the general term *item*. In chemical equilibrium problems, where molecules of different types play the roles of activities, the different kinds of atoms in the mixture are the items. The various items are denoted by a subscript i ($i = 1, 2, \ldots, M$).

In linear programming work, the quantity of an item required by an activity is assumed to be *proportional* to the quantity of activity level; if the item is not required, but produced, it is again assumed to be proportional to the quantity (or level) of the activity. The coefficient of proportionality is denoted by a_{ij}. The sign of a_{ij} depends on whether the item is required or produced by the activity. As we have already seen in § 3-3-(7), the sign convention used will be (+) if required and (−) if produced. Finally, b_i denotes the quantity of the i^{th} item made available to the program from outside (or exogenous) sources, if it is positive; it denotes the quantity to be produced by the program, if it is negative. The interdependencies between the x_j are expressed as a set of M linear equations such that the i^{th} equation gives a complete account of the i^{th} item. In general, this set of M linear equations is represented by

$$(1)\qquad \begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1N}x_N = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2N}x_N = b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{M1}x_1 + a_{M2}x_2 + \cdots + a_{MN}x_N = b_M \end{array}$$

where

$$(2)\qquad x_j \geq 0 \qquad j = (1, 2, \ldots, N)$$

We shall use capital M and N whenever we wish to refer to the *standard* form and use m and n for a *general* $m \times n$ system of linear equations. Any set of values x_j satisfying (1) and (2) is called a *feasible solution* because the corresponding schedule is possible or feasible.

The objective of a program, in practice, is often the most difficult part to express in mathematical terms. There are many historical reasons for this which go beyond the scope of this text. In many problems, however, the objective is simply one of carrying out the requirements (expressed by those b_i which are negative) in such a manner *that total costs are minimum.* Costs may be measured in dollars, in the number of people involved, or in a quantity of a scarce commodity used. In linear programming the total costs, denoted by z, are assumed to be a *linear function of the activity levels*:

$$(3)\qquad c_1x_1 + c_2x_2 + \cdots + c_Nx_N = z$$

DEFINITION: The linear form z is called the *objective function* (or form).

For purposes of solution we shall, as a rule, rewrite (3) as just another equation (4), where z is a *variable* whose value is to be minimized

$$(4)\qquad (-z) + c_1x_1 + c_2x_2 + \cdots + c_Nx_N = 0$$

In some problems the linear objective form is to be *maximized* rather than minimized. For example, the problem may be to produce the maximum dollar value of products under a fixed budget, fixed machine capacity, and fixed labor supply. Suppose the linear form expressing total profits to be maximized is

$$p_1x_1 + p_2x_2 + \cdots + p_Nx_N$$

This is obviously mathematically equivalent to minimizing

$$-p_1x_1 - p_2x_2 - \cdots - p_Nx_N = c_1x_1 + c_2x_2 + \cdots + c_Nx_N = z$$

where $c_1 = -p_1$, $c_2 = -p_2$, $\ldots$, $c_N = -p_N$.

Thus, the standard form of the linear programming problem is taken as the determination of a solution of a system of linear equations in non-negative variables which minimizes a linear form.

The Dual Linear Program. Associated with a linear program (called the *primal*) is another called the *dual*. The objective of the primal is to minimize, while that of the dual is to maximize. We have already seen in § 3-4 how the dual problem arises quite naturally as an algebraic check of a conjectured optimal solution to the primal problem. It will be noted that the array of coefficients and constants of the dual are obtained by transposing those of the primal. The variables π_i of the dual correspond to the equation i of the primal and are, however, unrestricted in sign. The variables x_j of the primal, restricted in sign to greater than or equal to zero, correspond to an inequality relation of "less than or equal" relations in the dual. For the general primal problem in standard form, the dual is to find values $\pi_1, \pi_2, \ldots, \pi_M$ and Max v satisfying

$$\begin{aligned}
a_{11}\pi_1 + a_{21}\pi_2 + \cdots + a_{M1}\pi_M &\leq c_1 \\
a_{12}\pi_1 + a_{22}\pi_2 + \cdots + a_{M2}\pi_M &\leq c_2 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\
a_{1N}\pi_1 + a_{2N}\pi_2 + \cdots + a_{MN}\pi_M &\leq c_N \\
b_1\pi_1 + b_2\pi_2 + \cdots + b_M\pi_M &= v \text{ (Max)}
\end{aligned}$$

In Chapter 4, a method is given for transforming the dual into a standard linear program. When a feasible solution to the primal is optimal the method for checking optimality gives rise automatically to an optimal solution for the dual problem. Since the dual of the dual problem is the primal problem, it is a matter of convenience whether one selects to solve the primal problem or the dual.

3-9. PROBLEMS

General.

1. (a) What is meant by the objective function; the central problem of linear programming; a feasible solution?
   (b) In many applications the variables and the equations each have a typical interpretation. What is it? Where do the inequality relations come in; the objective function?
   (c) Assuming that firing is the opposite of hiring, give reasons why it is better to treat these as two nonnegative activities rather than as a single activity with positive and negative activity levels.
2. If an activity such as steel production needs capital such as bricks and cement to build blast furnaces, what would the negative of these activities imply if they were used as admissible activities?
3. If the difference between production and requirements is interpreted as surplus or deficit (depending on sign), illustrate how surplus can be

interpreted as a storage activity and deficit as a purchasing activity in which all coefficients of the associated variables can be quite different.

Transportation Problem. (Refer to § 3-3. See § 3-4 for duality explanation.)

4. Two warehouses have canned tomatoes on hand and three stores require more in stock.

Warehouse	Cases on Hand	Store	Cases Required
I	100	A	75
II	200	B	125
		C	100

The cost (in cents) of shipping between warehouses and stores per case is given in the following table:

	A	B	C
I	10	14	30
II	12	20	17

(a) Set up the model describing the shipping of tomatoes from warehouses to stores, where the objective is to minimize the total shipping cost.

(b) Reformulate this problem assuming the cases required at B are only 60, and introducing a disposal activity at the warehouses at a loss of 5 cents per case disposed.

(c) Show that the optimal solution to problem (a) is the same if the cost per case from Warehouse I is increased by 3 cents; by 10 cents.

(d) Reformulate problem (a) assuming the cases available at Warehouse I are 90. Introduce a purchase activity from outside sources at a cost of 20 cents per case over the costs at Warehouses I and II.

(e) How would you formulate a model to include both the possibility of outside purchases at the destinations and disposal at the warehouses?

(f) State the dual of problems (b) and (d). How is the dual for (c) related to that for (a)?

5. Generalize problem 4 (a) for m warehouses and n destinations. Assume that the availability at the i source is a_i and requirement at the j^{th} destination is b_j. For part (a) assume $\sum_1^m a_i = \sum_1^n b_j$. Make the necessary modifications for parts (b), (c), and (d). The cost of transportation from source i to destination j is c_{ij}. Show in (a) there is one redundant equation. How does the deletion of one redundant equation affect the dual?

Blending Problem. (Refer to § 3-4.)

6. A housewife asks a butcher to grind up several cuts of beef to form a blend of equal parts of proteins and fats. The butcher, being conscientious,

wishes to do this at the least cost per pound of meat purchased exclusive of water content.

	Chuck	Flank	Porter-house	Rib Roast	Round	Rump	Sirloin
% Protein	19	20	16	17	19	16	17
% Fat	16	.18	25	23	11	28	20
Cost/lb	69	98	1.39	1.29	1.19	1.50	1.65

(a) What amounts of each type of meat should he use and how much should he charge?

(b) Usually he has extra fat available free per pound. How does this alter the solution?

(c) Solve the problem graphically.

(d) Find the dual.

7. (Thrall.)

(a) Suppose steaks contain per unit 1 unit of carbohydrates, 3 units of vitamins, 3 units of proteins and cost 50 units of cash. Suppose potatoes per unit contain 3, 4, 1, and cost 25 units of these items respectively. Letting x_1 be the quantity of steaks and x_2 the quantity of potatoes, express the mathematical relations that must be satisfied to meet the minimum requirements of 8 units of carbohydrates, 19 units of vitamins, and 7 units of protein. If x_1 and x_2 are to be chosen so that the cost of diet is a minimum, what is the objective function?

(b) Reduce the inequality system of (a) to an equality system in non-negative variables.

8. (a) Formulate the housewife problem of § 1-2.

(b) Is there any difference between the activity of inserting food i into the father's diet and the activity of inserting the same food into the children's diet?

(c) Could the housewife conceivably end up with the task of cooking five different dinners on the same day, one for each member of the family?

Product Mix Problem. (Refer to § 3-5.)

9. Solve the duals of the three primal problems within the product mix problem. How are the duals interrelated?

10. (a) Suppose contracts with various retailers have already been signed for the following quantities of desks:

Desk	1	2	3	4
Number sold	60	30	10	50

How does this affect the model?

(b) How does one interpret an optimal solution, if a fractional number of desks is obtained? One possible interpretation is that these are rates for a six-month period. Suppose the fractional solution is rounded to the nearest integer, find out how much change is required in the productivity coefficients or in the shop capacities for the adjusted solution to be optimal. Are the coefficients and constant terms really known accurately in any real situation?

11. A subcontractor has made arrangements to supply a company with 150 assemblies in January and 225 in February. Using an eight-hour shift the subcontractor can produce only 160 assemblies each month. By working the regular shift for two hours overtime, an additional 30 assemblies can be made, each with an overtime penalty of $20. Assemblies can be stored at a cost of $3 per month. Set up a model for finding the production program which minimizes costs.

12. A mass production house builder plans to build homes on 100 lots in a new subdivision. He has decided on 5 basic styles of homes: Ranch, Split-level, Colonial, Cape Cod, and Modern. To build the homes, the builder has two major contractors: masons for foundation work, and carpenters for the rest of the construction. The number of days required for the work is as follows:

	Ranch	Split-level	Colonial	Cape Cod	Modern
Foundation	1	2	2	1	1
Framework	4	7	6	5	3
Profit	2,000	3,000	2,500	1,700	2,000

The builder borrowed money at a very low interest rate for three years. Because it normally takes two months to sell a house, the builder wanted all homes to be completed in 34 months, or approximately 610 working days.

(a) How many of each style home should be built to maximize profit?

(b) If the builder wanted to build at least 10 of each style, what should be his building program to insure maximum profit?

(c) Solve by the method used for the product mix problem, § 3-5.

13. A machine problem of Kantorovich [1939-1].

Formulate, but do not solve. An assembled item consists of two metal parts. The milling work can be done on different machines, milling machines, turret lathes, or on automatic turret lathes. The basic data are available in the table at the top of p. 66. From this:

(a) Divide the work time of each machine to obtain the maximum number of completed items per hour.

(b) Prove that an optimal solution has the property that there will be no slack time on any of the machines; that equal numbers of each part will be made.

(c) State the dual of the primal problem.

Productivity of the Machines for Two Parts

Type of Machine	Number of Machines	Maximum Output* per Machine per Hour	
		First Part	Second Part
Milling machines	3	10	20
Turret lathes	3	20	30
Automatic turret lathes	1	30	80

* If devoted exclusively to making one of the parts.

14. (a) Generalize problem 13 to n machines, m parts, where the objective is to produce the largest number of completed assemblies.
    (b) Show, in general, if each machine is capable of making each part, and there is no value to the excess capacity of the machines or unmatched parts, any optimal solution will have only matched parts and will use all the machine capacity. What can happen if some machines are incapable of producing certain parts?
    (c) State the dual of the primal problem.
15. Suppose there are two types of assemblies instead of one and a "value" can be attached to each. Maximize the weighted output.
16. Extend the formulation of problems 14 and 15 to cover the following:
    (a) Suppose there is a limit on electricity used which depends on the task-machine combination.
    (b) Suppose it is possible, by the i^{th} mode of production, to produce $c_{i,k,l}$ units of the k^{th} part on the l^{th} machine.
    (c) Suppose it is possible to put values on surplus parts; on unused machine capacity.
17. Three parts can each be produced on two machines. Assume that there is no set-up time and that this is a continuous type production, that is, a part is first inserted in Machine 1 and then is immediately put in Machine 2 with practically no time elapsing between operations. The unit time per part in each machine and profit on each finished part is given by:

Machine	Part		
	A	B	C
1	.02	.03	.05
2	.05	.02	.04
Profit	.05	.04	.03

    (a) Formulate a model for the optimal product mix. Express this in terms of a linear inequality model, given that there are available only 40 hours on each machine. Transform the system into an equality system.

(b) Generalize to n different kinds of parts and m machines.

(c) State the dual of problems (a) and (b).

Simple Warehouse Problem. (Refer to § 3-6.)

18. (a) Reformulate the simple warehouse problem, § 3-6, if it is desired to have the quantities of selling, storing, and buying to be the same for the corresponding quarter each year. Formulate the yearly least-cost model assuming that the initial stock level is the same as the stock held in storage at the end of the year.

(b) Discuss the special properties of the coefficient matrix in a dynamic problem of this type.

On-the-Job Training Problem. (Refer to § 3-7.)

19. Reformulate the on-the-job training problem, § 3-7, assuming the cost of increasing the level of production above last week's level is $\$q = 4$ per unit of increase. There is no cost to decrease. All production is stored at a cost of $\$s = 1$ per unit per week until the last week. If the initial production level is $P_0 = 5$ and the final required inventory position is $g_5 = 200$ workers, what is an optimal production program?

20. A farmer may sell part of his crop and plant the remainder where his yield will be λ bushels per bushel planted. He expects to get p_1 dollars profit per bushel for the crop he has planted, p_2 and p_3 dollars per bushel for the two following crops. His first crop will be A bushels.

Problem: Set up the basic equations and the linear form which represents his total profits for the three periods. Show that it always pays to sell the third crop. Show that it pays to plant his entire first and second crop if $\lambda^2 p_3 > p_1$, $\lambda p_3 > p_2$. Show that it pays to sell the entire first crop if $p_1 - p_2 > 0$, $p_2 - p_3 > 0$. When does it pay to sell the entire second crop?

21. (Kemeny) The Chicken and Egg Problem.

Formulate: Suppose it takes a hen two weeks to lay 12 eggs for sale or to hatch 4. What is the best laying and hatching program if at the end of the fourth period all hens and chicks accumulated during the period are sold at 60 cents apiece and eggs at 10 cents apiece. Assume

(a) An initial inventory of 100 hens and 100 eggs,

(b) 100 hens and zero eggs,

(c) 100 hens and zero eggs and also a final inventory of 100 hens and zero eggs.

22. (Orchard-Hays.) A factory buys item A and produces item B. Each B requires one A and the factory has a production capacity of 3,000 B's per quarter year. However, A's are available in different amounts and B's are required in different amounts each quarter. Furthermore,

storing B's is expensive and the carryover of this item from one quarter to the next is to be minimized. At the beginning of the year, 3,000 A items are on hand and at least this many must be left over at the end of the year. The availability of A items and the requirement of B items by quarters is as follows:

1st quarter:	5,000	A's available,	1,000	B's required
2nd ,,	3,000	,,	4,000	,,
3rd ,,	1,000	,,	3,000	,,
4th ,,	2,000	,,	1,500	,,

There is storage room available for 10,000 A's or 2,000 B's or any combination in this ratio. Assume that, for each quarter q, the equation

$$s_q:\ A_q + 5B_q + S_q = 10{,}000 \qquad (q = 1, 2, 3, 4)$$

is sufficient to express the storage constraint (this ignores bottlenecks during a quarter). The variables are defined below.

Set up a linear programming model to minimize the carryover of item B each quarter subject to the stated restraints. For each quarter, use the following 7 variables:

M_q: amount of B items manufactured in quarter q.
p_q: amount of A items purchased in quarter q for use in quarter q (or later).
A_q: amount of A items unused at end of quarter q.
B_q: amount of B items on hand (over requirements) at end of quarter q.
C_q: excess production capacity during quarter q.
S_q: excess storage capacity during quarter q.
U_q: excess availability of A items during quarter q.

This gives 28 variables; one special one, for the end-of-year requirement on A's, is

v: excess inventory of A at end of year.

These 29 variables will have coefficients in 21 restraint equations as follows:

a_q: Balance equation in A items for quarter q.
b_q: Balance equation in B items produced in quarter q.
c_q: Production capacity restraint equation for quarter q.
p_q: Restraint equation for availability of A items in quarter q.
s_q: Balance equation for storage capacity in quarter q (see above).
v: Requirement equation for carryover of A items at end of year.

Find any feasible solution. Is your solution optimal? If the availability of A items is changed to 3,000 per quarter, what happens? Can you see how other changes in availability and requirement constants would make the problem harder? Impossible? Redundant? Inconsistent?

CHAPTER 4

LINEAR EQUATION AND INEQUALITY SYSTEMS

4-1. SYSTEMS OF EQUATIONS WITH THE SAME SOLUTION SET

Consistent Equations and Linear Combinations.

Because methods used for solving the linear programming problem depend on familiar methods for solving a system of linear equations, it is well at this point to review some elementary concepts. To facilitate the discussion, we consider the following example of two equations in three variables:

$$
(1)\qquad \begin{aligned} x_1 - x_2 + x_3 &= 2 \qquad (E_1)\\ 2x_1 + x_2 - x_3 &= 7 \qquad (E_2) \end{aligned}
$$

The ordered set of values $x_1 = 3$, $x_2 = 2$, $x_3 = 1$ is said to be a *solution* of E_1 because substitution of these values for x_1, x_2, x_3 in the first equation produces the identity, $2 = 2$. The solution (3, 2, 1) is said to *satisfy* equation E_1.

In general, suppose we have a system of m equations in n variables,

$$
(2)\qquad \begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}
$$

(This is an *arbitrary* $m \times n$ system of linear equations. We shall use capital M, N when we wish to designate the system of linear equations in the *standard* linear program arrived at in § 3-8-(1).) A *solution* of the i^{th} equation is an ordered set of numbers $(x_1', x_2', \dots, x_n')$ such that

$$
a_{i1}x_1' + a_{i2}x_2' + \dots + a_{in}x_n' = b_i
$$

An ordered set of numbers is said to be a *solution of a system* of equations provided it is a solution of each equation of the system. For example, because substituting (3, 2, 1) for the variables in equation E_2 yields $7 = 7$, an identity, we have (3, 2, 1) as a solution of both E_1 and E_2 and therefore of the system.

We usually speak of "a" solution rather than "the" solution to avoid questions of uniqueness. It is certainly evident that a system of equations need not possess a unique solution nor, indeed, any solution at all. Besides (3, 2, 1) for example, the system above is satisfied by any set of numbers of the form $(3, x_3' + 1, x_3')$ where x_3' may be chosen arbitrarily. A system which has solutions, unique or not, is called *consistent* or *solvable*. Otherwise, we refer to it as *inconsistent* or *unsolvable*.

The aggregate of solutions of a system is called its *solution set*. If the system is inconsistent its solution set is said to be *empty*.

Given a system such as (1) it is easy to construct new equations from it that have the property that every solution of (1) is also a solution of the new equation. In (3) the new equation is shown below the line; it is formed by multiplying the first equation by 2 and the second by -3, shown on the left, and summing.

$$\begin{array}{rl} 2: & x_1 - x_2 + x_3 = 2 \\ -3: & 2x_1 + x_2 - x_3 = 7 \\ \hline & -4x_1 - 5x_2 + 5x_3 = -17 \end{array} \tag{3}$$

It will be noted that the solution, (3, 2, 1), of the system (1) is also a solution of the new equation.

A scheme for generating new equations, whose solution set includes all the solutions of a general linear system (2), is shown in (4). For each equation i an arbitrary number, k_i, is chosen,[1] shown on the left; the new equation below the line is formed by multiplying the i^{th} equation by k_i and summing:

$$(4)\quad \left\{\begin{array}{rl} k_1: & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ k_2: & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ & \cdots\cdots\cdots\cdots\cdots\cdots \\ k_m: & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \\ \hline \end{array}\right.$$

$$d_1x_1 + d_2x_2 + \cdots + d_nx_n = d \tag{5}$$

The coefficients of the sum are easily read off; they are

$$(6)\quad \left\{\begin{array}{l} d_1 = k_1a_{11} + k_2a_{21} + \cdots + k_ma_{m1} \\ d_2 = k_1a_{12} + k_2a_{22} + \cdots + k_ma_{m2} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ d_n = k_1a_{1n} + k_2a_{2n} + \cdots + k_ma_{mn} \\ d\ \ = k_1b_1 + k_2b_2 + \cdots + k_mb_m \end{array}\right.$$

An equation such as (5) formed in this manner is called a *linear combination* of the original equations. The numbers k_i are called *multipliers* or *weights* of the linear combination.

[1] The constants k_i may be zero.

Writing (4) and (5) in *detached coefficient form* (7) and (8) we see that the operation of forming a linear combination of the equations corresponds to forming a linear combination of the *rows* of (7). By this we mean that we can form each element of row (8) by summing the products of k_i by the corresponding element in row i of (7).

$$(7)\qquad \begin{bmatrix} a_{11} & a_{22} \dots a_{1n} & b_1 \\ a_{21} & a_{22} \dots a_{2n} & b_2 \\ \dots & \dots & \dots \\ a_{m1} & a_{m2} \dots a_{mn} & b_m \end{bmatrix} \qquad \begin{matrix} \underline{\text{Multiplier}} \\ :k_1 \\ :k_2 \\ \\ :k_m \end{matrix}$$

$$(8)\qquad [\, d_1 \quad d_2 \ \dots d_n \quad d \,]$$

EXERCISE: Suppose a linear combination of the *columns* of (7) equals some other column. Show that this is true if the row in (8) is adjoined to those of (7).

Whenever a set of numbers $(x_1, x_2, \dots, x_n)$ constitutes a solution of (4), equation (5) becomes, upon substitution, a weighted sum of identities and hence an identity itself. Therefore, *every solution of a linear system is also a solution of any linear combination of the equations of the system.* Such an equation may therefore be inserted into a system of equations without affecting the solution set.

DEFINITIONS: If in a system of equations, an equation is a linear combination of the others, it is said to be *dependent* upon them; the dependent equation is called *redundant*. A *vacuous equation*, i.e., an equation of the form

$$0x_1 + 0x_2 + \dots + 0x_n = 0$$

is also called redundant when it occurs in a *single* equation system. A system containing no redundancy is called *independent.*

A linear system is clearly unsolvable or inconsistent if it is possible to exhibit a linear combination of the equations of the system of the form

$$(9)\qquad 0x_1 + 0x_2 + \dots + 0x_n = d \qquad \text{with } d \neq 0;$$

for any solution of the system would have to satisfy (9), but this is impossible no matter what values are assigned to the variables. We shall refer to (9) as an *inconsistent equation.* (See exercise below.) For example, the system

$$\begin{array}{rl} 1: & x_1 + x_2 + x_3 = 5 \\ -1: & x_1 + x_2 + x_3 = 4 \\ \hline & 0x_1 + 0x_2 + 0x_3 = 1 \end{array}$$

is unsolvable because the first equation states that a sum of three numbers is 5, while the second states that this same sum is 4. However, if we had proceeded to apply multipliers $k_1 = 1$, $k_2 = -1$, by way of eliminating, say, x_1, we would arrive automatically at the contradiction displayed below the equations. In general, the process of elimination applied to an inconsistent system will lead in due course to an inconsistent equation, as we shall show in the next section.

EXERCISE: Show that the only single-equation inconsistent linear system is of form (9).

EXERCISE: Show that if a system contains a vacuous equation, it is dependent.

How Systems Are Solved.

The usual "elimination" procedure for finding a solution of a system of equations is to *augment* the system by generating new equations by taking linear combinations in such a way that certain coefficients are zero. (This may be followed by the deletion of certain redundant equations.)

For example in (10) below, the equation E_1 is multiplied by $k_1 = -2$ and E_2 by $k_2 = 1$ so that upon summing the coefficient of x_1 vanishes. This yields equation E_3. These operations may be written symbolically, $E_3 = (-2)E_1 + (1)E_2$. Similarly we can form equation E_4 by multiplying E_3 by $\frac{1}{3}$ and we can form E_5 by adding E_4 to E_1. The *augmented system* $\{E_1, E_2, \ldots, E_5\}$ has the same solution set as the original system (1) because all equations such as E_4 and E_5 can be re-expressed as direct linear combinations of E_1 and E_2.

$$
(10) \qquad
\begin{array}{rrrrl}
x_1 - & x_2 + & x_3 = & 2 & \qquad (E_1) \\
2x_1 + & x_2 - & x_3 = & 7 & \qquad (E_2) \\
 & 3x_2 - & 3x_3 = & 3 & \qquad (E_3 = -2E_1 + E_2) \\
 & x_2 - & x_3 = & 1 & \qquad (E_4 = \tfrac{1}{3}E_3) \\
x_1 & & = & 3 & \qquad (E_5 = E_1 + E_4)
\end{array}
$$

It is interesting to note that the subsystem $\{E_4, E_5\}$ can be used to easily detect whether any equation is linearly dependent on it. Note that x_2 appears in E_4 with a unit coefficient and zero coefficient in E_5 and the opposite is true for x_1. This makes it easy to eliminate x_1 and x_2 from any other equation. For example, it is clear that if E_1 is to be a linear combination of E_4 and E_5 the multiplier of E_5 must be 1 and of E_4 must be -1. It is easily verified that $E_1 = E_5 - E_4$, $E_2 = 2E_5 + E_4$, $E_3 = 3E_4$. Thus all solutions of $\{E_4, E_5\}$ are also solutions of $\{E_1, E_2\}$, and as noted earlier, all solutions of $\{E_1, E_2\}$ are solutions of $\{E_4, E_5\}$; therefore *the solutions of the two subsystems are the same.*

A second advantage of $\{E_4, E_5\}$ is that it is easy to state the set of all possible solutions. Indeed, choose any arbitrary value for $x_3 = x_3^0$ and evaluate x_2 and x_1 in terms of x_3. In this case, $(x_1 = 3,\ x_2 = 1 + x_3^0,\ x_3 = x_3^0)$ describes the set of all solutions. For example, $x_3 = 0$ yields the *particular* solution $(x_1 = 3,\ x_2 = 1,\ x_3 = 0)$.

In general, the method of solving a system (we shall describe this in detail in § 4-2) is one of augmentation by linear combinations until in the enlarged system there is a subsystem whose solution set is easy to describe and such that each equation of the full system is linearly dependent upon it except possibly for the constant term. The subsystem arrived at belongs to a class called canonical.

DEFINITION: A *canonical* system with an ordered subset of variables, called *basic*, is a system such that for each i, the i^{th} basic variable has a unit coefficient in the i^{th} equation and has zero coefficients elsewhere.

For example, $\{E_4, E_5\}$ in (10) is canonical with x_2 associated with E_4 and x_1 with E_5. System (11) below is canonical because for each i, x_i has a unit coefficient in the i^{th} equation and zero elsewhere.

$$
\begin{array}{lllllr}
x_1 & & & + \bar{a}_{1,r+1}x_{r+1} + \cdots + \bar{a}_{1,n}x_n & = \bar{b}_1 \\
& x_2 & & + \bar{a}_{2,r+1}x_{r+1} + \cdots + \bar{a}_{2,n}x_n & = \bar{b}_2 \\
& & \ddots & \vdots & \vdots \\
& & & x_r + \bar{a}_{r,r+1}x_{r+1} + \cdots + \bar{a}_{r,n}x_n & = \bar{b}_r
\end{array}
\tag{11}
$$

EXERCISE: Show how by arbitrarily choosing values for $x_{r+1}, \ldots, x_n$ the class of all solutions can be generated. How can (11) be used to check easily whether or not another equation is dependent upon it?

Deletion of an equation that is a linear combination of the others is another operation that does not affect the solution set. If after an augmentation, one of the original equations in the system is found to be a linear combination of the others, it may be deleted. In effect the new equation becomes a "substitute" for one of the original equations. Where electronic computers are used, their limited capacity to store information makes this ability to throw away equations particularly important.

DEFINITION:[2] Two systems are called *equivalent* if one system may be derived from the other by inserting or by deleting a redundant equation or if one system may be derived from the other through a chain of systems each linked to its predecessor by such an insertion or deletion.

THEOREM 1: *Equivalent systems have the same solution set.*

Elementary Operations.

There are two simple but important types of linear combinations which may be used to obtain equivalent systems.

1. Replacing any equation, E_t, by the equation $[kE_t]$ with $k \neq 0$.
2. Replacing any equation, E_t, by the equation $[E_t + kE_i]$ where E_i is any other equation of the system.

To prove an elementary operation of the first type results in an equivalent

[2] This definition of equivalence is due to A. W. Tucker (verbal communication).

system, insert kE_t as a new equation after E_t, then delete E_t. Note that E_t is a redundant equation for it can be formed from kE_t by $1/k[kE_t]$ if $k \neq 0$. Similarly, for the second type, insert $E_t + kE_i$ after E_t and then delete E_t. Note that E_t is a redundant equation, for it is given by $[E_t + kE_i] - kE_i$.

One way to transform our example (1) into the equivalent system (10) by a sequence of elementary operations is given below:

		Elementary Operation
$x_1 - x_2 + x_3 = 2$	(E_1)	
$2x_1 + x_2 - x_3 = 7$	(E_2)	
		Replace E_2 by $E'_2 = E_2 + E_1$
$x_1 - x_2 + x_3 = 2$	(E_1)	
$3x_1 = 9$	(E'_2)	
		Replace E_1 by $E'_1 = E_1 - \frac{1}{3}E'_2$
$-x_2 + x_3 = -1$	(E'_1)	
$3x_1 = 9$	(E'_2)	
		Replace E'_1 by $E''_1 = -E'_1$
$x_2 - x_3 = 1$	(E''_1)	
$3x_1 = 9$	(E'_2)	

In general, corresponding to each elementary operation there is an *inverse* operation which is also elementary and of the same type. For example, starting with the last pair of equations, we can obtain the next to last pair by replacing E''_1 by $E'_1 = -E''_1$; then we can obtain the second pair from it in turn by replacing E'_1 by $E_1 = E'_1 + \frac{1}{3}E'_2$ and then the first pair by replacing E'_2 by $E_2 = E'_2 - E_1$.

THEOREM 2: *Corresponding to a sequence of elementary operations is an inverse sequence of elementary operations by which a given system can be obtained from the derived system.*

We can also see that if a system can be derived from a given system by a sequence of elementary operations it implies that it is possible to obtain each row of the derived system in detached coefficient form directly by a linear combination of the rows of the given system. Conversely, by Theorem 2, each row of the given system is some linear combination of the rows of the derived system.

THEOREM 3: *The rows of two equivalent systems in detached coefficient form can be obtained one from the other by linear combinations.*

THEOREM 4: *If the t^{th} equation of a given system is replaced by a linear combination with multipliers k_i where $k_t \neq 0$, an equivalent system is obtained.*

EXERCISE: Prove Theorems 2, 3, 4.

The most important property of systems derived by elementary operations is, by Theorem 1, that they have the *same solution set.*

An interesting question now arises. Are all linear equation systems with the same solution set obtainable by a sequence of inserting and deleting of redundant equations? We shall show (and this is the substance of § 4-2, Theorem 1 and § 8-1, Theorem 4) that if two systems have the same solution set and are solvable, then they are equivalent. On the other hand, if the systems are not solvable, this is not necessarily the case. Indeed, consider the two systems

$$\{0x = 1\} \quad \text{and} \quad \left\{\begin{matrix} 0x = 1 \\ 1x = 1 \end{matrix}\right\};$$

both have empty, hence identical, solution sets. It is obvious that if these two systems were equivalent some multiple (linear combination) of the equation $0x = 1$ of the first system would yield the equation $1x = 1$ of the second. This is clearly impossible.

4-2. CANONICAL SYSTEMS

Solving Square Systems.

The systems of linear equations dealt with in high school algebra courses commonly have exactly as many equations as variables; in the general system (2) of the preceding section this would be the case when $m = n$. Such a system of equations is called a square system.

Assuming a square system of m equations in m unknowns possesses a unique solution, the usual process of solving such a system consists in eliminating one unknown, setting aside one equation, and working with a reduced system having one less equation and one less unknown. The process is repeated a total of $m - 1$ times, resulting in a single equation with one variable. Its value is then substituted in the preceding equation to determine the value of another variable. This process, called "back solution," is repeated until all the variables are evaluated [Gauss, 1826-1]. Our immediate purpose is to review this procedure in detail to show that it is in fact nothing more than a sequence of elementary operations that replaces the original system by an equivalent system in simple *diagonal form* (1). Here the solution set is evident.

$$\begin{aligned} x_1 \qquad\qquad\quad &= \bar{b}_1 \\ x_2 \qquad\quad &= \bar{b}_2 \\ \ddots \qquad &\quad \vdots \\ x_m &= \bar{b}_m \end{aligned} \tag{1}$$

Consider a system of 3 equations in 3 unknowns:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned} \tag{2}$$

If $a_{11} \neq 0$, then the first equation can be used to eliminate x_1 from the second equation by the elementary operation $E_2' = E_2 - (a_{21}/a_{11})E_1$, and to eliminate x_1 from the third equation by the elementary operation on the resulting system $E_3' = E_3 - (a_{31}/a_{11})E_1$. Thus we obtain an equivalent system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{22}'x_2 + a_{23}'x_3 &= b_2' \\ a_{32}'x_2 + a_{33}'x_3 &= b_3' \end{aligned} \qquad (a_{11} \neq 0) \tag{3}$$

The top equation is normally set aside and the process repeated with the reduced system. If $a_{22}' \neq 0$ then the second equation can be used to eliminate x_2 from the third equation, resulting in the equivalent *triangular system:*

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{22}'x_2 + a_{23}'x_3 &= b_2' \\ a_{33}''x_3 &= b_3'' \end{aligned} \qquad (a_{11} \neq 0,\ a_{22}' \neq 0) \tag{4}$$

If $a_{33}'' \neq 0$, the back solution begins by solving for x_3 in the last equation. Then one substitutes x_3 into the second equation to evaluate x_2. Finally, both values are substituted into the first equation. These two substitutions amount to exactly the same thing as using the third equation to eliminate x_3 from the second and first equations by the successive elementary operations $E_2'' = E_2' - (a_{23}'/a_{33}'')E_3''$ and $E_1' = E_1 - (a_{13}/a_{33}'')E_3''$, resulting in

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 \qquad &= b_1' \\ + a_{22}'x_2 \qquad &= b_2'' \\ + a_{33}''x_3 &= b_3'' \end{aligned} \qquad (a_{11} \neq 0,\ a_{22}' \neq 0,\ a_{33}'' \neq 0) \tag{5}$$

Substituting the value of x_2 obtained from the second equation into the first to evaluate x_1 has the same effect as the elementary operation $E_1'' = E_1 - (a_{12}/a_{22}')E_2''$ and yields

$$\begin{aligned} a_{11}x_1 \qquad\qquad &= b_1'' \\ a_{22}'x_2 \qquad &= b_2'' \\ a_{33}''x_3 &= b_3'' \end{aligned} \qquad (a_{11} \neq 0,\ a_{22}' \neq 0,\ a_{33}'' \neq 0) \tag{6}$$

Finally, division by the diagonal coefficients, which is a sequence of three successive elementary operations, yields a diagonal system of the form (1).

If the system possesses a *unique* solution it will always be possible to carry out this process, but not always in the order indicated. Thus, if $a_{1s} = 0$, for example, one may pass to any other term whose coefficient is non-zero, say $a_{ts}x_s$, called the *pivot*, for the elimination of x_s.

In this case the t^{th} equation may be used to eliminate x_s from the other equations by a sequence of elementary operations, replacing the i^{th} equation by the sum of the i^{th} equation, and the t^{th} equation multiplied by $-a_{is}/a_{ts}$. If this process is repeated on each reduced system obtained by setting aside the equation used for the elimination, this will result finally in a system corresponding to (1) and (4) which can be put into diagonal and triangular forms by suitable rearrangement of the order of the equations.

In general a square system will be said to be ***triangular*** if upon suitable rearrangement of its rows and its variables, all coefficients below the diagonal are zero and all coefficients on the diagonal are non-zero; if, in addition, only the diagonal coefficients are non-zero, it is called *diagonal*.

As an example of reduction to triangular and diagonal forms, consider the 3×3 system

$$\begin{aligned} \text{I}_0\colon\quad & x_1 + x_2 + x_3 = 1 \\ \text{II}_0\colon\quad & x_1 - x_2 + x_3 = 3 \\ \text{III}_0\colon\quad & x_1 + 2x_2 - x_3 = 4 \end{aligned}$$

It can be reduced to triangular form as follows:

		Operation
I_1:	$x_1 + x_2 + x_3 = 1$	$\text{I}_1 = \text{I}_0$
II_1:	$-2x_2 = 2$	$\text{II}_1 = \text{II}_0 - \text{I}_0$
III_1:	$x_2 - 2x_3 = 3$	$\text{III}_1 = \text{III}_0 - \text{I}_0$

		Operation
I_2:	$x_1 + x_2 + x_3 = 1$	$\text{I}_2 = \text{I}_1$
II_2:	$x_2 = -1$	$\text{II}_2 = -\frac{1}{2}\text{II}_1$
III_2:	$x_3 = -2$	$\text{III}_2 = -\frac{1}{2}(\text{III}_1 + \frac{1}{2}\text{II}_1)$

This last system, $(\text{I}_2, \text{II}_2, \text{III}_2)$, is triangular and can readily be reduced to the diagonal form,

		Operation
I_3:	$x_1 = +4$	$\text{I}_3 = \text{I}_2 - \text{II}_2 - \text{III}_2$
II_3:	$x_2 = -1$	$\text{II}_3 = \text{II}_2$
III_3:	$x_3 = -2$	$\text{III}_3 = \text{III}_2$

in which the solution is explicit.

A Pivotal Reduction of a General System.

Instead of a square system suppose, more generally, we have a system of m equations in n variables, with $m \leq n$,

(7)
$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array}$$

We are interested in ways of replacing this system, if possible, by an equivalent *canonical* system (see § 4-1 for definition). In this form the solution set is evident and it is easy to detect whether or not any other system is equivalent to it.

(8)

Canonical system with basic variables $x_1, x_2, \ldots, x_m$		
x_1	$+ \bar{a}_{1,m+1}x_{m+1} \cdots + \bar{a}_{1j}x_j + \cdots + \bar{a}_{1n}x_n$	$= \bar{b}_1$
x_2	$+ \bar{a}_{2,m+1}x_{m+1} \cdots + \bar{a}_{2j}x_j + \cdots + \bar{a}_{2n}x_n$	$= \bar{b}_2$
$\ddots$	$\vdots$	$\vdots$
x_m	$+ \bar{a}_{m,m+1}x_{m+1} \cdots + \bar{a}_{mj}x_j + \cdots + \bar{a}_{mn}x_n$	$= \bar{b}_m$
Basic Variables	Non-basic (or Independent) Variables	Constants

The diagonal system (1) for square systems is clearly a special case of the canonical system (8) with $m = n$. The standard procedure for reducing (if possible) a general system (7) to equivalent canonical form will now be discussed.

The principles are best illustrated with an example and then generalized. Consider the 2×4 system,

$$\begin{array}{rl} + \mathbf{x_2} + x_3 + x_4 &= 5 \\ x_1 - 2x_2 - x_3 + x_4 &= 2 \end{array}$$

Choose as "pivot element" any term with non-zero coefficient such as the *boldfaced* term in the first equation, and eliminate the corresponding variable, x_2, from the other equations by means of elementary operations.

$$\begin{array}{rl} x_2 + x_3 + x_4 &= 5 \\ + \mathbf{x_1} \quad + x_3 + 3x_4 &= 12 \end{array}$$

Next choose as pivot any term in the remaining equations such as the boldfaced term in the second equation above. Eliminate the corresponding variable, in this case x_1, from all the other equations. (Because x_1 happens to have a zero coefficient in the first equation, no further eliminations are in fact required.) Hence, rearranging the equations gives the system

$$\begin{array}{rl} x_1 \quad + x_3 + 3x_4 &= 12 \\ x_2 + x_3 + x_4 &= 5 \end{array}$$

From this canonical system with basic variables x_1, x_2 it is evident, by setting $x_3 = x_4 = 0$, that one solution is $x_1 = 12$, $x_2 = 5$, $x_3 = x_4 = 0$.

Pivoting.

Definition: A *pivot operation* consists of m elementary operations which replace a system by an equivalent system in which a specified variable has a coefficient of unity in one equation and zero elsewhere. The detailed steps are as follows:

(a) Select a term $a_{rs}x_s$ in system (7) such that $a_{rs} \neq 0$, called the *pivot term.*

(b) Replace the r^{th} equation by the r^{th} equation multiplied by $(1/a_{rs})$.

(c) For each $i = 1, 2, \ldots, m$ except $i = r$, replace the i^{th} equation by the sum of the i^{th} equation and the replaced r^{th} equation multiplied by $(-a_{is})$.

In general the reduction to some canonical form can be accomplished by a sequence of pivot operations. For the first pivot term select any term $a_{rs}x_s$ such that $a_{rs} \neq 0$. After the first pivoting, the second pivot term is selected using a non-zero term from any equation except r, say equation r'. After pivoting, the third pivot term is selected in the resulting m-equation system from any equation except r and r', say equation r''. In general, repeat the pivoting operation, always choosing the pivot term from equations that do not correspond to equations previously selected. Continue in this manner, terminating either when m pivots have been used or when, after selecting r variables, it is not possible to find a non-zero term in any equation except those corresponding to previously selected pivot terms.

For example, if the successive pivoting was done on variables $x_1, x_2, \ldots, x_r$ in the corresponding equations $i = 1, 2, \ldots, r$, then the original system (7) would be reduced to an equivalent system of form (9), which we will refer to as the *reduced system* with pivotal variables $x_1, x_2, \ldots, x_r$. We shall also refer to a system as reduced relative to r pivotal variables if, by changing the order of the variables and equations, it can be put into form (9).

(9) Reduced system with pivotal variables $x_1, x_2, \ldots, x_r$

$$\begin{array}{lllllll}
x_1 & & & + \bar{a}_{1,r+1}x_{r+1} + \bar{a}_{1,r+2}x_{r+2} + \cdots + \bar{a}_{1n}x_n & = \bar{b}_1 \\
 & x_2 & & + \bar{a}_{2,r+1}x_{r+1} + \bar{a}_{2,r+2}x_{r+2} + \cdots + \bar{a}_{2n}x_n & = \bar{b}_2 \\
 & & \ddots & \vdots & \vdots \\
 & & x_r & + \bar{a}_{r,r+1}x_{r+1} + \bar{a}_{r,r+2}x_{r+2} + \cdots + \bar{a}_{rn}x_n & = \bar{b}_r \\
\hline
 & & & 0 \cdot x_{r+1} + \cdots\cdots + 0 \cdot x_n & = \bar{b}_{r+1} \\
 & & & \cdots\cdots\cdots\cdots & \\
 & & & 0 \cdot x_{r+1} + \cdots\cdots + 0 \cdot x_n & = \bar{b}_m
\end{array}$$

Since (9) was obtained from (7) by a sequence of pivoting operations each of which consists of m elementary operations, it follows that the reduced system is (a) formed from *linear combinations* of the original system, and (b) *equivalent* to the original system.

The original system (7) is solvable if and only if its reduced system (9) is solvable, and (9) is solvable if and only if

$$\bar{b}_{r+1} = \bar{b}_{r+2} = \cdots = \bar{b}_m = 0 \tag{10}$$

If (10) holds, the solution set is immediately evident because any values of the (independent) variables $x_{r+1}, \ldots, x_n$ determine corresponding values for the (dependent) variables $x_1, \ldots, x_r$. On the other hand if $\bar{b}_{r+i} \neq 0$ for some i, the solution set is *empty* because the $(r + i)^{\text{th}}$ equation is inconsistent for it states that $0 = \bar{b}_{r+i}$. In this case the original system (7) and the reduced system (9) are both inconsistent (unsolvable).

Canonical System.

If the original system is consistent, the system formed by dropping the vacuous equations from the reduced system is called its *canonical equivalent* with the pivotal variables as basic.

(11)

Canonical system with basic variables $x_1, x_2, \ldots, x_r$

$$\begin{array}{llllll} x_1 & & & + \bar{a}_{1,r+1}x_{r+1} + \bar{a}_{1,r+2}x_{r+2} + \cdots + \bar{a}_{1n}x_n & = & \bar{b}_1 \\ & x_2 & & + \bar{a}_{2,r+1}x_{r+1} + \bar{a}_{2,r+2}x_{r+2} + \cdots + \bar{a}_{2n}x_n & = & \bar{b}_2 \\ & & \ddots & \vdots & & \vdots \\ & & x_r & + \bar{a}_{r,r+1}x_{r+1} + \bar{a}_{r,r+2}x_{r+2} + \cdots + \bar{a}_{rn}x_n & = & \bar{b}_r \end{array}$$

Dependent (basic) Variables	Independent Variables	Constants

Uniqueness of a Canonical Equivalent.

The fundamental property of a canonical system resulting from the reduction process is that for any other system with the same solution set a reduction can be effected using the same pivotal variables and the resulting canonical system will be *identical* if the equations are reordered so that their correspondence with the basic variables is the same in both systems.

THEOREM 1: *There is at most one equivalent canonical system with a fixed set of basic variables.*

PROOF: Let there be two equivalent canonical systems relative to $x_1, x_2, \ldots, x_r$. Substituting $x_{r+1} = x_{r+2} = \cdots = x_n = 0$ into the first system, we get $x_1 = \bar{b}_1, x_2 = \bar{b}_2, \ldots, x_r = \bar{b}_r$. Because of equivalence, substitution into the second system should yield the same values; this will only be true if their respective constant terms are the same. Similarly, substituting the values for independent variables of $x_{r+1} = x_{r+2} = \cdots = x_n = 0$, except $x_s = 1$, will show (after equating constant terms) that their corresponding coefficients of x_s are the same for any $s = r + 1, r + 2, \ldots, n$.

The above theorem can also be established by applying

LEMMA 1: *Any equation can either be generated by a unique linear combination of the equations of a canonical system (the weights being the coefficients of the basic variables in the equation) or no linear combination exists.*

EXERCISE: Apply the lemma to test whether a system is equivalent to a canonical system.

Basic Solutions.

The special solution obtained by setting the independent variables equal to zero and solving for the dependent variables is called a basic solution. Thus if (8) is the canonical system of (7) with basic variables $x_1, x_2, \ldots, x_m$, the corresponding basic solution is

$$(12) \quad x_1 = \bar{b}_1, x_2 = \bar{b}_2, \ldots, x_m = \bar{b}_m; x_{m+1} = x_{m+2} = \ldots = x_n = 0$$

Degenerate Solutions.

A basic solution is degenerate if the values of one or more of the dependent (basic) variables are zero. In particular, the basic solution (12) is degenerate if $\bar{b}_i = 0$ for at least one i.

Basis.

In accordance with the special usage in linear programming, the term *basis* refers to the *ordered set* of columns of the original *independent* system (in detached coefficient form) corresponding to the ordered set of basic variables of a canonical equivalent. The columns of the basis will be called *basic columns* (or *basic activities*).

In the example following (8) the basis associated with the canonical system with basic variables x_2, x_1 is $\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$.

The reader is referred to § 8-1, "Pivot Theory," which extends the results of this section. A proof is given there that solvable systems with identical solution sets are equivalent.

4-3. LINEAR INEQUALITIES

In the remaining sections of this chapter we shall turn our attention to linear inequality systems which also play an important role in the solution of linear programs.

Just as in the special case of solving linear equations, it is possible that there exist no solutions to a system of linear inequalities, or there may exist many. To see this geometrically, let us take the linear programming is often alternatively stated as that of *minimizing a linear form subject to a system of linear inequalities.*

Reduction of Linear Inequality Systems to Standard Form.

By a linear inequality is meant a relation of the form

$$a_1x_1 + a_2x_2 + \dots + a_nx_n \leq b \tag{1}$$

rather than a *strict* linear inequality

$$a_1x_1 + a_2x_2 + \dots + a_nx_n < b \tag{2}$$

It should be noted that if a system includes strict inequalities, it is not always possible to find values for the variables which satisfy the inequalities and at the same time minimize a linear form. For example, there is no value of $x_1 > 1$ which minimizes the form $z = x_1$.

Any problem involving a system of linear inequalities can be transformed into another system in standard form, i.e., into a system of equations in nonnegative variables by one of several devices. Steps (A) and (B) below constitute one method; Steps (A) and (B′) below constitute a second method. In Chapter 6 the dual is developed for a system of linear inequalities; it will be noted that the dual system is in standard form. This constitutes a third way. The first method and perhaps the easiest is:

Step (A). Change any linear inequality restraint, such as (1), to an equation by adding a slack variable $x_{n+1} \geq 0$, thus

$$a_1x_1 + a_2x_2 + \dots + a_nx_n + x_{n+1} = b \tag{3}$$

Step (B). Noting that any number can be written as the difference of two positive numbers, replace any variable x_j not restricted in sign by the difference of two nonnegative variables

$$x_j = x'_j - x''_j \qquad (x'_j \geq 0,\ x''_j \geq 0) \tag{4}$$

Exercise: (Tucker) Prove in place of (4) that each x_j unrestricted in sign can be replaced by $x'_j - x_0$ where $x'_j \geq 0$ and where $x_0 \geq 0$ is the same for all such j.

Step (B′). As an alternative to Step (B), let x_j be any variable not restricted in sign that appears in the k^{th} equation with a non-zero coefficient. Solve the equation for x_j and substitute its value in the remaining equations if any and in the objective form z. Setting the equation aside, the remaining modified equations (if any) constitute a reduced system of constraints. The procedure is repeated with the new linear programming problem until either: (i) a reduced system of constraints is obtained in which all remaining variables are nonnegative, or (ii) no equations remain in the reduced system.

Once a solution to the reduced problem is obtained, a solution to the original problem is obtained by successive substitutions, in reverse order, in the eliminated equations.

To justify the procedure, note first that the minimum for the reduced system is less than or equal to that of the full system, because it involves only part of the conditions of the problem. On the other hand, the solution obtained for the full system (by the reverse substitution) has the same value for z and is, therefore, minimum.

EXAMPLE 1: Transform the system into standard form.

$$\begin{aligned} x_1 + x_2 &\geq 6 \\ x_1 + 2x_2 &= z \end{aligned} \tag{5}$$

Step (A). Introduce slack variable x_3.

$$\begin{aligned} x_1 + x_2 - x_3 &= 6 \\ x_1 + 2x_2 \qquad &= z \end{aligned} \qquad (x_3 \geq 0) \tag{6}$$

Step (B). Substitute $x_1 = x_1' - x_1''$, $x_2 = x_2' - x_2''$, obtaining

$$\begin{aligned} (x_1' - x_1'') + (x_2' - x_2'') - x_3 &= 6 \\ (x_1' - x_1'') + 2(x_2' - x_2'') \qquad &= z \end{aligned} \qquad (x_3 \geq 0,\ x_j' \geq 0,\ x_j'' \geq 0) \tag{7}$$

Step (B'). Solve the equation $x_1 + x_2 - x_3 = 6$ for x_1, which is unrestricted in sign, obtaining

$$x_1 = 6 - x_2 + x_3 \qquad (x_3 \geq 0) \tag{8}$$

and substitute in the objective form z to get

$$x_2 + x_3 + 6 = z \qquad (x_3 \geq 0) \tag{9}$$

We now solve for x_2, but no equations remain for substitution; this is case (ii). A general solution to the original system (5) can be obtained by choosing any value for $x_3 \geq 0$, any value for z, and substituting these values in (9) and then in (8) to determine x_2 and x_1. Notice that no finite lower bound for z exists since z may be chosen arbitrarily.

EXAMPLE 2: Transform the system

$$\begin{aligned} -x_1 - x_2 &\leq -6 \\ -x_1 + x_2 &\geq 5 \\ x_1 + 2x_2 &= z \end{aligned} \tag{10}$$

into standard form.

Step (A). Introduce slack variables x_3 and x_4

$$\begin{aligned} -x_1 - x_2 + x_3 \qquad &= -6 \\ -x_1 + x_2 \qquad - x_4 &= 5 \\ x_1 + 2x_2 \qquad &= z \end{aligned} \qquad (x_3 \geq 0,\ x_4 \geq 0) \tag{11}$$

Step (B). Substitute $x_1' - x_1''$, $x_2' - x_2''$ for x_1, x_2 where $x_j' \geq 0$, $x_j'' \geq 0$; or

Step (B'). Solve the first equation for x_1 and substitute in the second equation and the z-form. Next, solve the modified second equation for x_2 and substitute in the modified z-form. This eliminates the constraint equations and we are left with a reduced system consisting of only one constraint in nonnegative variables x_3, x_4:

$$z = (23 + 3x_3 + x_4)/2 \qquad (x_3, x_4 \geq 0) \tag{12}$$

and the eliminated equations

$$\begin{aligned} x_1 &= 6 - x_2 + x_3 \\ x_2 &= (11 + x_3 + x_4)/2 \end{aligned} \tag{13}$$

A general solution to the original system of constraints is obtained by selecting any $x_3 \geq 0$, $x_4 \geq 0$, and determining x_2 and x_1 from (13). If the objective is to minimize z, then, from (12), the optimum solution is found by setting $x_3 = 0$, $x_4 = 0$, obtaining $z = \frac{23}{2}$, $x_2 = \frac{11}{2}$, $x_1 = \frac{1}{2}$.

In general, suppose we have n inequalities in $k \leq n$ variables $(u_1, u_2, \ldots, u_k)$ which are unrestricted in sign, and a form z in these variables to be minimized:

$$\begin{aligned} \alpha_{j1}u_1 + \alpha_{j2}u_2 + \cdots + \alpha_{jk}u_k - \alpha_{j0} &\geq 0 \qquad (j = 1, 2, \ldots, n) \\ \gamma_1 u_1 + \gamma_2 u_2 + \cdots + \gamma_k u_k \qquad &= z \end{aligned}$$

where α_{ji} and γ_i are constants. If we set

$$x_j = \alpha_{j1}u_1 + \alpha_{j2}u_2 + \cdots + \alpha_{jk}u_k - \alpha_{j0} \qquad (j = 1, 2, \ldots, n)$$

then clearly

$$x_j \geq 0 \qquad (j = 1, 2, \ldots, n)$$

If we assume that it is possible to solve at least one set of k of the equations for $u_1, u_2, \ldots, u_k$ in terms of the x_j, then the substitution of these values of u_i in the remaining equations and the z-form yields $n - k$ equations and a z-form in nonnegative variables. Thus under this assumption, *n inequalities in $k \leq n$ variables is equivalent to $m = n - k$ equations in n nonnegative variables.*

Reduction of an Equation System to an Inequality System.

Conversely, any problem involving equations can be replaced by an equivalent system involving only linear inequality restraints. One way is to replace each equation

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \tag{14}$$

by the two inequalities,

$$\begin{aligned} a_1x_1 + a_2x_2 + \cdots + a_nx_n &\geq b \\ a_1x_1 + a_2x_2 + \cdots + a_nx_n &\leq b \end{aligned} \tag{15}$$

Another way is to change each equation into an inequality ($\geq$) and to change the sum of the equations into the opposite inequality. Thus, we may replace the equation system § 4-1-(2) by

$$(16)\qquad \begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \geq b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \geq b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \geq b_m \end{cases}$$

$$(\textstyle\sum a_{i1})x_1 + \cdots + (\sum a_{in})x_n \leq \sum b_i$$

4-6. PROBLEMS

Systems of Equations with the Same Solution Set. (Refer to § 4-1.)

1. *Review.* What are the elementary operations? Why do they lead to an equivalent system of equations?
2. In a transportation problem without slack variables, show that the system of equations in § 3-3 is redundant. Interpret this as a redundancy relation among items. Show any equation is redundant. Is redundancy possible with slack variables? Write out the equations in Tableau Form for both cases.
3. Reduce

$$\begin{aligned} 3x_1 + 2x_2 + x_3 &= 4 \\ x_1 + x_2 + x_3 &= 3 \end{aligned}$$

to

$$\begin{aligned} 2x_1 + x_2 \qquad &= 1 \\ x_1 \qquad - x_3 &= -2 \end{aligned}$$

by elementary operations. What solution is evident from the reduced system? Check this solution by substituting in the original system.

4. Prove that it is not possible to transform systems § 4-1-(1) by elementary operations to equivalent form $\{E_4, E_5\}$ of § 4-1-(10) without using at least one elementary operation of the first type.

Canonical Systems. (Refer to § 4-2.)

5. Prove that (except possibly for sign) the product of the pivots equals the determinant of the basis where the basis is the square matrix of coefficients of the basic set of variables.
6. *Review.* What is the canonical form? Can every system of equations be reduced to canonical form?
7. *Review.* What constitutes a basic set of variables? What is the basic solution associated with the basic set? What is a degenerate basic solution?

8. What are the independent or non-basic variables?
9. Need a basic solution be feasible, i.e., are the values of variables associated with a basic solution necessarily nonnegative?
10. What elementary operations can be used to transform

$$\begin{cases} 2x_1 + x_2 + x_3 = 6 \\ x_1 + x_2 + x_3 = 4 \\ 2x_1 + 3x_2 + x_3 = 8 \end{cases} \quad \text{into} \quad \begin{cases} x_1 \qquad\qquad\quad = 2 \\ \qquad x_2 \qquad\quad = 1 \\ x_1 + 3x_2 + x_3 = 6 \end{cases}$$

Can you find a solution to this system? Now reduce this system to canonical form.

11. Put the following system in canonical form with x_1 and x_4 as basic variables.

$$\begin{aligned} x_1 - 2x_2 + x_3 \qquad &= 1 \\ x_1 + x_2 \qquad + x_4 &= 4 \end{aligned}$$

12. Reduce the system

$$\begin{aligned} 5x_1 - 4x_2 + 13x_3 - 2x_4 + x_5 &= 20 \\ x_1 - x_2 + 5x_3 - x_4 + x_5 &= 8 \end{aligned}$$

to canonical form using variables x_2 and x_4 as basic variables.

13. Reduce the system below to canonical form with respect to variables x_2 and x_4 if possible and find the associated basic solution.

$$\begin{aligned} 2x_1 + 3x_2 + 4x_3 + 5x_4 &= -1 \\ 2x_1 - 3x_2 - x_3 - x_4 &= -7 \end{aligned}$$

14. Consider the system

$$\begin{aligned} 3x_1 + 2x_2 + 11x_3 + 5x_4 - 3x_5 &= 5 \\ x_1 + x_2 + 4x_3 + 3x_4 + x_5 &= 2 \end{aligned}$$

   (a) Reduce this system to canonical form using x_1 and x_2 as basic variables. What solution is suggested by this canonical form when variables x_3, x_4, x_5 are all zero?
   (b) Reduce the original system to canonical form with x_1 and x_3 as basic variables. What solution is suggested by this canonical form?
   (c) Now, using the results of (a), find the canonical form of (b) without referring to the original system of equations.

15. Consider the system

$$\begin{aligned} 2x_1 + 3x_2 - 2x_3 - 7x_4 &= 1 \\ x_1 + x_2 + x_3 + 3x_4 &= 6 \\ x_1 - x_2 + x_3 + 5x_4 &= 4 \end{aligned}$$

   (a) Reduce this to a canonical system with x_1, x_2, and x_3 as basic variables. What solution is suggested by this canonical form? Check by substitution into the original system.

(b) From the canonical form of (a) find another canonical form with x_1, x_2, and x_4 as basic variables. What is the solution when $x_3 = 0$?

(c) From the canonical form of (b) find the canonical form with x_1, x_3, and x_4 as basic variables. What is the solution when $x_2 = 0$?

(d) From the canonical form of (c) find the canonical form with x_2, x_3, and x_4 as basic variables. What is the solution when $x_1 = 0$?

16. In the system below the variables y_1, y_2, y_3 are expressed in terms of x_1, x_2, x_3. Re-express the values of x_1, x_2, x_3 in terms of y_1, y_2, y_3 and show that the resulting system is equivalent to the original system. Show that the original system is essentially in canonical form with respect to y_1, y_2, y_3 while the resulting system is in canonical form with respect to x_1, x_2, x_3.

$$\begin{aligned} 2x_1 + 3x_2 + 4x_3 &= y_1 \\ x_1 - x_2 + x_3 &= y_2 \\ 4x_1 + 3x_2 + 2x_3 &= y_3 \end{aligned}$$

17. The system expressing y_1, y_2, y_3 in terms of x_1, x_2, x_3 is called the *inverse system*. Why is the inverse unique? Show, in general, that if there are m equations that express $y_1, y_2, \ldots, y_m$ in terms of $x_1, x_2, \ldots, x_m$, the inverse system expressing $x_1, x_2, \ldots, x_m$ in terms of $y_1, y_2, \ldots, y_m$ exists if $x_1, x_2, \ldots, x_m$ is a basic set of variables.

18. *Review.* Why are two equivalent canonical systems with respect to the same basic variables identical?

19. Why is it not possible to have two or more different basic solutions relative to a given set of basic variables?

Linear Inequalities. (Refer to § 4-3.)

20. Reduce each of the inequality systems (a), (b), and (c) to an equivalent system of equations with nonnegative variables by two different methods.

(a) $x_1 + 2x_2 \geq 3$, $x_1 - 2x_2 \geq -4$, $x_1 + 7x_2 \leq 6$

(b) $x_1 + x_2 \geq 2$, $x_1 - x_2 \leq 4$, $x_1 + x_2 \leq 7$

(c) $x_1 + x_2 \geq 2$, $x_1 - x_2 \leq 4$, $x_1 + x_2 + x_3 \leq 7$

Show that systems (b) and (c) correspond to cases (i) and (ii) of the alternate (B′). Show how to construct the class of all solutions for (c).

21. Transform the system of equations in nonnegative variables into a system of inequalities:

$$\begin{aligned} 2x_1 + 3x_2 + 4x_3 &= 5 \qquad (x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0) \\ 4x_1 - 7x_2 + 3x_3 &= 4 \end{aligned}$$

22. Show that no lower bound for z exists

(a) for the system $x_1 \geq 0$, $-x_1 = z$;

(b) for the system

$$x_1 - x_2 = 1 \qquad (x_1 \geq 0,\ x_2 \geq 0)$$
$$-x_1 - x_2 = z$$

(c) Show that a lower bound for z exists for the system $x_1 > 0$, $x_1 = z$, but while there are *feasible* solutions, there exists no *optimal* feasible solution.

23. Suppose (a_{ij}, b_i, c_j) denote the coefficients and constants before reduction to canonical form with respect to $x_1, x_2, \ldots, x_m$, and $(\bar{a}_{ij}, \bar{b}_i, \bar{c}_j)$ denote the coefficients and constants after reduction. In the dual of the original system,

$$\sum_1^m a_{ij}\pi_i \leq c_j \qquad (j = 1, 2, \ldots, n)$$

$$\sum_1^m b_i\pi_i = v \text{ (Max)}$$

introduce slack variables $y_j \geq 0$ and eliminate the unrestricted variables π_i by using pivots in the first m of the n equations. Show that the result is the standard linear program in n nonnegative variables and $n - m$ equations, and results in

$$\begin{cases} \displaystyle\sum_1^m \bar{a}_{ij}y_i + y_j = \bar{c}_j & (j = m + 1, \ldots, n) \\ \displaystyle\sum_1^m \bar{b}_i y_i = v - \sum_1^m \bar{b}_i c_i \end{cases}$$

24. Use the "center of gravity method" of Chapter 3 to find $x_j \geq 0$ and Min z satisfying

$$\begin{aligned} z &= 1x_1 + 2x_2 + 3x_3 + 4x_4 \\ 4 &= x_1 + x_2 + x_3 + x_4 \\ -2 &= 1x_1 - 2x_2 + 3x_3 - 4x_4 \end{aligned}$$

25. Reduce the system

$$\begin{aligned} x_1 + x_2 + x_3 &= 5 \qquad (x_1 \geq 0,\ x_3 \geq 0) \\ x_1 - x_2 + x_3 &= 7 \\ x_1 + 2x_2 + 4x_3 &= 2 \end{aligned}$$

to an equivalent inequality system.

26. Solve graphically the system in nonnegative variables:

$$\begin{aligned} x_1 + x_2 &\leq 1 \\ 4x_1 + 8x_2 &\leq 32 \\ x_1 + x_2 &\leq 4 \\ x_1 - 2x_2 &\geq 2 \end{aligned}$$

What inequalities are implied by others?

Fourier-Motzkin Elimination Method. (Refer to § 4-4.)

27. Using the Fourier-Motzkin Elimination Method, find values of x_1, x_2, and z satisfying Problem 29, Case (c), and yielding Min $z = x_2$.
28. Use the Elimination Method to solve for nonnegative x_i and Min z satisfying the system

$$\begin{aligned} x_1 + x_2 &\geq 1 \\ x_1 + x_2 &\leq 2 \\ x_1 - x_2 &\leq 1 \\ x_1 - x_2 &\geq -1 \\ - x_2 &= z \end{aligned}$$

Graph and show the convex set of feasible solutions. Modify the z form in four different ways, so that the solution is not unique.

Linear Programs in Inequality Form. (Refer to § 4-5.)

29. Discuss, by graphing, whether there exists zero, one, or many solutions to a system of inequalities in the following cases:

Case (a)	Case (b)	Case (c)	Case (d)
$x_1 \geq 0$	$x_1 \geq 0$	$x_1 \geq 0$	$x_1 \geq 0$
$x_2 \geq 0$	$x_2 \geq 0$	$x_2 \geq 0$	$x_2 \geq 0$
$x_1 + x_2 \geq 2$	$x_1 + x_2 \geq 2$	$x_1 + x_2 \geq 2$	$x_1 + x_2 \geq 2$
	$x_1 + 2x_2 \leq 6$	$x_1 + 2x_2 \leq 6$	$x_1 + 2x_2 \leq 6$
	$-x_1 + 4x_2 \geq 0$	$-x_1 + 4x_2 \geq 0$	$-x_1 + 4x_2 \geq 0$
		$-x_1 + x_2 \geq 2$	$-x_1 + x_2 \geq 2$
			$-x_1 + x_2 \geq 3$

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CHAPTER 5

THE SIMPLEX METHOD

Outline: The standard form for the central problem of linear programming, as developed in § 3-8-(1), consists in finding values for a set of nonnegative variables that satisfies a system of linear equations and minimizes a linear form z.

We distinguish between the ***simplex method*** which starts with a linear program in standard form and the ***simplex algorithm*** which starts with a canonical form, consists of a sequence of pivot operations, and forms the main *subroutine* of the simplex method.

The first step of the simplex method is the introduction into the standard form of certain *artificial variables*. The resulting auxiliary problem is in canonical form. At this point the simplex algorithm is employed. It consists of a sequence of pivot operations referred to as Phase I that produces a succession of different canonical forms. The objective is to find a feasible solution if one exists. If the final canonical form yields such a solution, the simplex algorithm is again applied in a second succession of pivot operations referred to as Phase II. The objective is to find an optimal feasible solution if one exists.

In § 5-1 that follows the simplex algorithm will be described; its use, as part of the simplex method, will be developed in § 5-2.

5-1. SIMPLEX ALGORITHM

The simplex algorithm is always initiated with a program whose equations are in canonical form; for example, let us suppose we have canonical system (1), (2) with basic variables $x_1, x_2, \ldots, x_m, (-z)$.[1] The relation of this m-equation n-variable canonical system to the M-equation, N-variable system of the standard form will become clear in § 5-2.

Problem: Find values of $x_1 \geq 0,\ x_2 \geq 0,\ \ldots,\ x_n \geq 0$ and Min z satisfying

$$
\begin{array}{lllllll}
(1) & x_1 & & + \bar{a}_{1,m+1} x_{m+1} + \cdots + \bar{a}_{1j} x_j + \cdots + \bar{a}_{1n} x_n = \bar{b}_1 \\
& & x_2 & + \bar{a}_{2,m+1} x_{m+1} + \cdots + \bar{a}_{2j} x_j + \cdots + \bar{a}_{2n} x_n = \bar{b}_2 \\
& & \ddots & \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
& & x_m & + \bar{a}_{m,m+1} x_{m+1} + \cdots + \bar{a}_{mj} x_j + \cdots + \bar{a}_{mn} x_n = \bar{b}_m \\
(2) & & (-z) & + \quad \bar{c}_{m+1} x_{m+1} + \cdots + \quad \bar{c}_j x_j + \cdots + \quad \bar{c}_n x_n = -\bar{z}_0
\end{array}
$$

[1] That is to say $-z$ is treated as a basic variable. In the literature the reader may find this variable labeled x_0 and equation (2) arranged ahead of those of (1).

where $\bar{a}_{ij}$, $\bar{c}_j$, $\bar{b}_i$, and $\bar{z}_0$ are constants. In this canonical form the basic solution is

(3) $z = \bar{z}_0;\ x_1 = \bar{b}_1;\ x_2 = \bar{b}_2, \ldots, x_m = \bar{b}_m;\ x_{m+1} = x_{m+2} = \ldots = x_n = 0$

Since it is *assumed* that this basic solution is also *feasible*, the values of the x_j in (3) are nonnegative, so that

(4) $$\bar{b}_1 \geq 0, \bar{b}_2 \geq 0, \ldots, \bar{b}_m \geq 0$$

DEFINITION: If (4) holds, we say that the linear program is presented in *feasible canonical form.*

Test for Optimality.

We have seen that the canonical form can provide an immediate evaluation of the associated basic solution. It may also be used to determine whether the basic solution (if feasible) is minimal, through an examination of the coefficients of the "modified" objective equation (2).

DEFINITION: The coefficients, $\bar{c}_j$, in the cost or objective form of the canonical system (2), are called *relative cost factors*—"relative" because their values will depend on the choice of the basic set of variables.

THEOREM 1: *A basic feasible solution is a minimal feasible solution with total cost $\bar{z}_0$ if all relative cost factors are nonnegative:*

$$\bar{c}_j \geq 0 \qquad (j = 1, 2, \ldots, n)$$

PROOF: Referring to the canonical form, it is obvious that if the coefficients of the modified cost form are all positive or zero, the smallest value of the sum $\Sigma\bar{c}_j x_j$ is zero for *any choice* of nonnegative x_j. Thus, the smallest value of $z - \bar{z}_0$ is zero and Min $z \geq \bar{z}_0$. In the particular case of the basic feasible solution, we have $z = \bar{z}_0$; hence Min $z = \bar{z}_0$ and the solution is optimal. It is also clear that

THEOREM 2: *Given a minimal basic feasible solution with relative cost factors $\bar{c}_j \geq 0$, then any other feasible solution (not necessarily basic) with the property that $x_j = 0$ for all $\bar{c}_j > 0$ is also a minimal solution; moreover, a solution with the property that $x_j > 0$ and $\bar{c}_j > 0$ for some j cannot be a minimal solution.*

COROLLARY: *A basic feasible solution is the unique minimal feasible solution if $\bar{c}_j > 0$ for all non-basic variables.*

Improving a Non-optimal Basic Feasible Solution: An Example.

To illustrate, consider the problem of minimizing z where

(5) $$\begin{aligned} 5x_1 - 4x_2 + 13x_3 - 2x_4 + x_5 &= 20 \\ x_1 - x_2 + 5x_3 - x_4 + x_5 &= 8 \\ x_1 + 6x_2 - 7x_3 + x_4 + 5x_5 &= z \end{aligned} \qquad (x_j \geq 0)$$

Let us assume we know that x_1, x_5, and $(-z)$ can be used as basic variables and that the basic solution will be feasible. Accordingly, we can reduce system (5) to equivalent canonical form relative to x_5, x_1, $(-z)$:

$$
\begin{array}{lrcl}
(6) & -\tfrac{1}{4}x_2 + 3x_3 - \tfrac{3}{4}x_4 + x_5 & = & 5 \\
 & x_1 - \tfrac{3}{4}x_2 + \mathbf{2x_3} - \tfrac{1}{4}x_4 & = & 3 \\
 & 8x_2 - 24x_3 + 5x_4 \qquad -z & = & -28
\end{array}
$$

except that we have not bothered to rearrange the order of the variables and equations. The meaning of the boldfaced term will be discussed later. The basic feasible solution to (6) is immediately,

$$(7) \qquad x_1 = 3,\ x_5 = 5,\ x_2 = x_3 = x_4 = 0,\ z = 28$$

Note that an arbitrary pair of variables will not necessarily yield a basic solution to (5) which is feasible. For example, had the variables x_1 and x_2 been chosen as basic variables, the basic solution would have been

$$x_1 = -12,\ x_2 = -20,\ x_3 = x_4 = x_5 = 0,\ z = -132$$

which is not feasible since x_1 and x_2 are negative.

For the numerical example (4), one relative cost factor of its canonical form, (6), is negative, namely -24, the coefficient of x_3. The optimality test of Theorem 1 thus fails. If x_3 is increased to any positive value (the other non-basic variables remaining zero), it is evident that the value of z would be reduced because the corresponding value of z is given by

$$(8) \qquad z = 28 - 24x_3$$

It seems reasonable, therefore, to try to make x_3 as large as possible, since the larger the value of x_3, the smaller will be the value of z. Now the value of x_3 cannot be increased indefinitely while the other non-basic variables remain zero, because the corresponding values of the basic variables satisfying (6) are

$$
\begin{array}{lrcl}
(9) & x_5 & = & 5 - 3x_3 \\
 & x_1 & = & 3 - 2x_3
\end{array}
$$

and we see that if x_3 increases beyond $\tfrac{3}{2}$, then x_1 becomes negative, and that if x_3 increases beyond $\tfrac{5}{3}$, x_5 also becomes negative. Obviously, the largest permissible value of x_3 is the smaller of these, namely $x_3 = \tfrac{3}{2}$, which yields upon substitution in (8) and (9) a new feasible solution with lower cost:

$$(10) \qquad x_3 = \tfrac{3}{2},\ x_5 = \tfrac{1}{2},\ x_1 = x_2 = x_4 = 0,\ z = -8$$

This solution reduces z from 28 to -8; our immediate objective is to discover whether or not it is a minimal solution. This time a short cut is possible. A new canonical form with new basic variables, x_3 and x_5, can be obtained directly from the old canonical form with x_1 and x_5 basic. Choose as *pivot term* that x_3 term which limited the maximum amount that the basic variables, x_1 and x_5, could be adjusted without becoming negative, namely the boldfaced term, $2x_3$. Eliminating with respect to x_3, the new canonical form relative to x_5, x_3 and $(-z)$ becomes

$$
\begin{aligned}
&-\tfrac{3}{2}x_1 + \tfrac{7}{8}x_2 \qquad\ \ - \tfrac{3}{8}x_4 + x_5 \qquad\ = \tfrac{1}{2}\\
&\ \ \tfrac{1}{2}x_1 - \tfrac{3}{8}x_2 + x_3 - \tfrac{1}{8}x_4 \qquad\qquad\ = \tfrac{3}{2}\\
&\ 12x_1 - \ x_2 \qquad\quad + 2x_4 \qquad\ - z = 8
\end{aligned}
\tag{11}
$$

This gives the basic feasible solution, (10). Although the value of z has been reduced, the coefficient $\bar{c}_2 = -1$ indicates that the solution still is not minimal and that a better solution can be obtained by increasing the value of x_2, keeping the other non-basic variables, $x_1 = x_4 = 0$, and solving for new values for x_5, x_3, and z in terms of x_2:

$$
\begin{aligned}
x_5 &= \tfrac{1}{2} - \tfrac{7}{8}x_2\\
x_3 &= \tfrac{3}{2} + \tfrac{3}{8}x_2\\
z &= -8 - x_2
\end{aligned}
\tag{12}
$$

Note that the second equation places no bound on the increase of x_2, but that the first equation restricts x_2 to a maximum of $(1/2) \div (7/8)$ which reduces x_5 to zero. Therefore, the *pivot term*, $\tfrac{7}{8}x_2$ in the first equation of (11), is used for the next elimination. The new set of basic variables is x_2 and x_3. Reducing system (11) to canonical form relative to x_2, x_3, $(-z)$ gives

$$
\begin{aligned}
&-\tfrac{12}{7}x_1 + x_2 \qquad\ - \ \tfrac{3}{7}x_4 + \tfrac{8}{7}x_5 \qquad\ = \tfrac{4}{7}\\
&-\tfrac{1}{7}x_1 \qquad\ + x_3 - \ \tfrac{2}{7}x_4 + \tfrac{3}{7}x_5 \qquad\ = \tfrac{12}{7}\\
&\ \ \tfrac{72}{7}x_1 \qquad\qquad\quad + \tfrac{11}{7}x_4 + \tfrac{8}{7}x_5 - z = \tfrac{60}{7}
\end{aligned}
\tag{13}
$$

and the basic feasible solution

$$
x_2 = \tfrac{4}{7},\ x_3 = \tfrac{12}{7},\ x_1 = x_4 = x_5 = 0,\ z = -\tfrac{60}{7}
\tag{14}
$$

Since all relative cost factors for the non-basic variables are positive, this solution is the unique minimal solution by the corollary of Theorem 2. This optimal solution was found from our initial basic solution (7) in two iterations.

Improving a Non-optimal Basic Feasible Solution in General.

As we have seen in the numerical example, the canonical form provides an immediate criterion for testing the optimality of a basic feasible solution. Furthermore, if the criterion is not satisfied, another solution is generated which reduces the value of the cost or objective function (except for certain degenerate cases).

Let us now formalize this procedure of improving a non-optimal basic feasible solution. If at least one relative cost factor, $\bar{c}_j$, in the canonical form (2) is negative, it is possible, assuming non-degeneracy (all $b_i > 0$), to construct a new basic feasible solution with a total cost lower than $z = \bar{z}_0$. The lower cost solution can be obtained by increasing the value of one of the non-basic variables, x_s, and adjusting the values of the basic variables accordingly, where x_s is any variable whose relative cost factor $\bar{c}_s$ is negative. In particular, the index s can be chosen such that

$$\bar{c}_s = \text{Min } \bar{c}_j < 0 \tag{15}$$

This is the rule for choice of s followed in practical computational work because it is convenient and because it has been found that it usually leads to fewer iterations of the algorithm than just choosing for s any j such that $\bar{c}_j < 0$.

Using the canonical form (1) and (2), we construct a solution in which x_s takes on some positive value, the values of all other non-basic variables are still zero, and the values of the basic variables, including z, are adjusted to take care of the increase in x_s:

$$\begin{aligned} x_1 &= \bar{b}_1 - \bar{a}_{1s}x_s \\ x_2 &= \bar{b}_2 - \bar{a}_{2s}x_s \\ &\cdots\cdots\cdots\cdots \\ x_m &= \bar{b}_m - \bar{a}_{ms}x_s \end{aligned} \tag{16}$$

$$z = \bar{z}_0 + \bar{c}_s x_s \qquad (\bar{c}_s < 0) \tag{17}$$

Since $\bar{c}_s$ has been chosen negative, it is clear that the value of x_s should be made as large as possible in order to make the value of z as small as possible. The only thing that prevents our setting x_s infinitely large is the possibility that the value of one of the basic variables in (16) will become negative. However, if all $\bar{a}_{is} \leq 0$, then x_s can be made arbitrarily large, establishing:

THEOREM 3: *If in the canonical system, for some s, all coefficients $\bar{a}_{is}$ are nonpositive and $\bar{c}_s$ is negative, then a class of feasible solutions can be constructed where the set of z values has no lower bound.*

On the other hand, if at least one $\bar{a}_{is}$ is positive, it will not be possible to increase the value of x_s indefinitely, because, whenever $x_s > \bar{b}_i/\bar{a}_{is}$, the value of x_i must be negative. If $\bar{a}_{is}$ is positive for more than one value of i, then the smallest of such ratios, whose row subscript will be denoted by r, will determine the largest value of x_s possible under the nonnegativity assumption. The greatest value for x_s permissible under the assumption will be

$$x_s^* = \frac{\bar{b}_r}{\bar{a}_{rs}} = \underset{\bar{a}_{is}>0}{\text{Min}} \frac{\bar{b}_i}{\bar{a}_{is}} \geq 0 \qquad (\bar{a}_{rs} > 0) \tag{18}$$

where it should be particularly noted that *only* those i and r are considered for which $\bar{a}_{is} > 0$, $\bar{a}_{rs} > 0$. The choice of r in case of a tie is arbitrary unless among those tied, $\bar{b}_i = 0$; in the latter (degenerate) case r may be chosen at random (with equal probability) from among them. For example, if

$\bar{a}_{1s} > 0$ and $\bar{a}_{2s} > 0$ but $\bar{b}_1 = \bar{b}_2 = 0$, then one may flip a coin to decide whether $r = 1$ or $r = 2$.[2]

The basic solution is degenerate if the values of one or more of the basic variables are zero (see § 4-2). In this case it is clear by (16) that, if for some $\bar{a}_{is} > 0$, it happens that the corresponding value $\bar{b}_i$ of the basic variable is zero, then no increase in x_s is possible that will maintain nonnegative values of the basic variables and therefore z will not decrease. However, if the basic solution is *nondegenerate* we have:

THEOREM 4: *If in the canonical system for some s the relative cost factor $\bar{c}_s$ is negative and at least one other coefficient $\bar{a}_{is}$ is positive, then from a nondegenerate basic feasible solution a new basic feasible solution can be constructed with lower total cost z.*

Specifically, we shall show that the replacing of x_r by x_s in the set of basic variables $x_1, x_2, \ldots, x_m$, results in a new set that is basic, and a corresponding basic solution that is feasible. We shall show feasibility first. Substituting the value of $x_s^* \geq 0$ determined by (18) into (16) and (17) gives a feasible solution,

$$\begin{array}{lll} x_i = \bar{b}_i - \bar{a}_{is}x_s^* \geq 0 & & (i = 1, 2, \ldots, m; i \neq r) \\ x_s = x_s^* & \text{where} & x_s^* = \bar{b}_r/\bar{a}_{rs} \geq 0 \\ x_j = 0 & & (j = r, m+1, \ldots, n; j \neq s) \end{array} \tag{19}$$

with total cost

$$z = \bar{z}_0 + \bar{c}_s x_s^* \leq \bar{z}_0 \qquad (\bar{c}_s < 0) \tag{20}$$

This feasible solution is different from the previous one since $\bar{b}_r \neq 0$ by assumption; $x_s^* > 0$ and $z < \bar{z}_0$.

It remains to be shown that the new feasible solution is basic. It is clear, from the definition in (18) of the index r, that

$$x_r = \bar{b}_r - \bar{a}_{rs}x_s^* = 0 \tag{21}$$

We are trying to show that x_s and $x_1, x_2, \ldots, x_m$ (excluding x_r) constitute a new basic set of variables. To see this, we simply observe that since $\bar{a}_{rs} > 0$, we may use the r^{th} equation of (1) and $\bar{a}_{rs}$ as pivot element to eliminate the variable, x_s, from the other equations and the minimizing form. Only this one elimination is needed to reduce the system to canonical form relative to the new set of variables. This fact constitutes the key to the computational efficiency of the simplex method. The new basic solution is unique by § 4-2, Theorem 1; hence its values are given by (19).

[2] The choice of r in case of a tie has been the subject of much investigation because of the theoretical possibility that a poor choice could lead to a repetition of the same basic solution after a number of iterations. For practical work an arbitrary choice may be used—W. Orchard-Hays [1956-1] who has experimented with various procedures, reports fewer iterations often result in practical problems using $i = r$ with maximum denominator $\bar{a}_{is}$ among those tied. (See § 6-1 and Chapter 10.)

Iterative Procedure.

The new basic feasible solution can be tested again for optimality by $\bar{c}_s = \text{Min } \bar{c}_j \geq 0$. If it is not optimal, then one may choose by criterion (15) a new variable, x_s, to increase and proceed to construct either: (a) a class of solutions in which there is no lower bound for z (if all $\bar{a}_{is} \leq 0$), or (b) a new basic feasible solution in which the cost z is lower than the previous one (provided the values of the basic variables for the latter are strictly positive; otherwise the new value of z may be equal to the previous value).

The simplex algorithm consists of repeating this cycle again and again, terminating only when there has been constructed either

(a) a class of feasible solutions for which $z \rightarrow -\infty$ or

(b) an optimal basic feasible solution (all $\bar{c}_j \geq 0$).

THEOREM 5: *Assuming nondegeneracy at each iteration, the simplex algorithm will terminate in a finite number of iterations.*

PROOF: There is only a finite number of ways to choose a set of m basic variables out of n variables. If the algorithm were to continue indefinitely, it could only do so by repeating the same basic set of variables—hence, the same canonical system and the same value of z. (See Uniqueness Theorem, § 4-2, Theorem 1.) This repetition cannot occur since the value of z decreases with each iteration.

When degenerate solutions occur, we can no longer argue that the procedure will necessarily terminate in a finite number of iterations, because under degeneracy it is possible for $\bar{b}_r = 0$ in (19), in which case the value of z decreases a *zero* amount in (20) and it is conceivable that the same basic set of variables may recur. If one were to continue, with the same selection of s and r for each iteration as before, the same basic set would recur after, say, k iterations, and again after $2k$ iterations, etc., indefinitely. There is therefore the possibility of circling (cycling)[3] in the simplex algorithm. In fact, examples have been constructed to show that this can happen; see Chapter 10.

We have shown the convergence of the simplex method to an optimal solution in a finite number of iterations only for the case of nondegenerate basic solutions. In § 6-1 we will justify the random choice rule, and in Chapter 10 we will show a simple way to change (perturb) the constant terms slightly, so as to assure nondegeneracy. We will prove that the procedure given there is valid even under degeneracy.

5-2. THE TWO PHASES OF THE SIMPLEX METHOD

The Problem.

The standard form, developed in Chapter 3, for the central mathematical problem of linear programming consists of finding values for $x_1, x_2, \ldots, x_N$ satisfying the simultaneous system of equations,

[3] In the literature the term "cycling" is used [Hoffman, 1951-1; Beale, 1952-1]. To avoid possible confusion with the term "cycle," which we use synonymously with "iteration," we have adopted "circling."

$$
\begin{aligned}
(1) \qquad & a_{11}x_1 + a_{12}x_2 + \ldots + a_{1N}x_N = b_1 \\
& a_{21}x_1 + a_{22}x_2 + \ldots + a_{2N}x_N = b_2 \\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
& a_{M1}x_1 + a_{M2}x_2 + \ldots + a_{MN}x_N = b_M
\end{aligned}
$$

and minimizing the objective form

$$(2) \qquad c_1x_1 + c_2x_2 + \ldots + c_Nx_N = z$$

where the x_j are restricted to be nonnegative:

$$(3) \qquad x_j \geq 0 \qquad (j = 1, 2, \ldots, N)$$

The simplex method is in general use for solving this problem. The method employs the simplex algorithm presented in § 5-1 in two phases which will be described in this section.

Many problems encountered in practice often have a starting feasible canonical form readily at hand. For example, one can immediately construct a great variety of starting basic feasible solutions for the important class called "transportation" problems; see Chapter 14. Economic models often contain storage and slack activities, permitting an obvious starting solution in which nothing but these activities takes place. Such a solution may be a long way from the optimum solution, but at least it is an easy start. Usually little or no effort is required in these cases to reduce the problem to canonical form. When this is the case, the Phase I procedure referred to above will not be necessary.

Other problems encountered in practice do not provide an obvious starting feasible canonical form. This is true when the model does not have slack variables for some equations, or when the slack variables have negative coefficients. Nothing may be known (mathematically speaking) about the problem. It may have

(a) *Redundancies:* This could occur, for example, if an equation balancing money flow had been obtained from the equations balancing material flows by multiplying price by quantity and summing. The classic transportation problem provides a second example (see § 3-3; see also the blending problem, § 3-4, for a third case).

(b) *Inconsistencies:* This could be caused by outright clerical errors, the use of inconsistent data, or by the specification of requirements which cannot be filled from the available resources. For example, one may pose a problem in which resources are known to be in short supply, and the main question is whether or not a feasible solution exists.

It is clear that a general mathematical technique must be developed to solve linear programming problems free of any prior knowledge or assumptions about the systems being solved. In fact, if there are inconsistencies or redundancies, these are important facts to be uncovered.

The Phase I procedure uses the simplex algorithm itself to provide a starting feasible canonical form (if it exists) for Phase II. It has several important features.

(a) No assumptions are made regarding the original system; it may be redundant, inconsistent, or not solvable in nonnegative numbers.
(b) No eliminations are required to obtain an initial solution in canonical form for Phase I.
(c) The end product of Phase I is a basic feasible solution (if it exists) in canonical form ready to initiate Phase II.

Outline of the Procedure.

A. Arrange the original system of equations so that all constant terms b_i are positive or zero by changing, where necessary, the signs on both sides of any of the equations.

B. Augment the system to include a basic set of *artificial* or *error variables* $x_{N+1} \geq 0$, $x_{N+2} \geq 0, \ldots, x_{N+M} \geq 0$, so that it becomes

(4)

$$\begin{array}{llllll}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1N}x_N & + x_{N+1} & & & & = b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2N}x_N & & + x_{N+2} & & & = b_2 \quad (b_i \geq 0) \\
\quad\vdots & & & \ddots & & \quad\vdots \\
a_{M1}x_1 + a_{M2}x_2 + \cdots + a_{MN}x_N & & & & + x_{N+M} & = b_M \\
c_1x_1 + c_2x_2 + \cdots + c_Nx_N & & & & + (-z) & = 0
\end{array}$$

and

(5) $$x_j \geq 0 \quad (j = 1, 2, \ldots, N, N+1, \ldots, N+M)$$

C. (Phase I): Use the simplex algorithm (with no sign restriction on z) to find a solution to (4) and (5) which minimizes the sum of the artificial variables, denoted by w:

(6) $$x_{N+1} + x_{N+2} + \cdots + x_{N+M} = w$$

Equation (6) is called the *infeasibility form*. The initial feasible canonical system for Phase I is obtained by selecting as basic variables $x_{N+1}, x_{N+2}, \ldots, x_{N+M}$, $(-z)$, $(-w)$ and eliminating these variables (except w) from the w form by subtracting the sum of the first M equations of (4) from (6), yielding

(7)

Admissible Variables	Artificial Variables				
$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1N}x_N$	$+x_{N+1}$				$= b_1$
$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2N}x_N$		$+x_{N+2}$			$= b_2$
$\vdots$			$\ddots$		$\vdots$
$a_{M1}x_1 + a_{M2}x_2 + \cdots + a_{MN}x_N$				$+x_{N+M}$	$= b_M$
$c_1x_1 + c_2x_2 + \cdots + c_Nx_N$				$-z$	$= 0$
$d_1x_1 + d_2x_2 + \cdots + d_Nx_N$				$-w$	$= -w_0$

where $b_i \geq 0$ and

(8)
$$d_j = -(a_{1j} + a_{2j} + \ldots + a_{Mj}) \quad (j = 1, 2, \ldots, N)$$
$$-w_0 = -(b_1 + b_2 + \ldots + b_M)$$

Writing (7) in detached coefficient form constitutes the *initial tableau* for Phase I (see Table 5-2-I).

D. If Min $w > 0$, then no feasible solution exists and the procedure is terminated. On the other hand, if Min $w = 0$, initiate Phase II of the simplex algorithm by (i) dropping from further consideration all non-basic variables x_j whose corresponding coefficients $\bar{d}_j$ are positive (not zero) in the final modified w-equation; (ii) replacing the linear form w (as modified by various eliminations) by the linear form z, after first eliminating from the z-form all basic variables. (In practical computational work the elimination of the basic variables from the z-form is usually done on each iteration of Phase I; see Tables 5-2-I, 5-2-II, and 5-2-III. If this is the case, then the modified z-form may be used immediately to initiate Phase II.)

E. (Phase II): Apply the simplex algorithm to the adjusted feasible canonical form at end of Phase I to obtain a solution which minimizes the value of z or a class of solutions such that $z \to -\infty$.

The above procedure for Phase I deserves some discussion. It is clear that if there exists a feasible solution to the original system (1) then this same solution also satisfies (4) and (5) with the artificial variables set equal to zero; thus, $w = 0$ in this case. From (6), the smallest possible value for w is zero since w is the sum of nonnegative variables. Hence, if feasible solutions exist, the minimum value of w will be $w = 0$; conversely, if a solution is obtained for (4) and (5) with $w = 0$, it is clear that all $x_{N+i} = 0$ and the values of x_j for $j \leq N$ constitute a feasible solution to (1). It also follows that if Min $w > 0$, then no feasible solutions to (1) exist.

Whenever the original system contains redundancies and often when degenerate solutions occur, artificial variables will remain as part of the basic set of variables in Phase II. Thus, it is necessary that their values in Phase II never exceed zero. This is accomplished in D above where all non-basic variables are dropped whose relative cost factors for w are positive. To see this we note that the w form at the end of Phase I satisfies

(9)
$$\bar{d}_1 x_1 + \bar{d}_2 x_2 + \ldots + \bar{d}_{M+N} x_{M+N} = w - \bar{w}_0$$

where $\bar{d}_j \geq 0$ and $\bar{w}_0 = 0$, if feasible solutions exist. For feasibility, w must be zero, which means that every x_j corresponding to $\bar{d}_j > 0$ must be zero; hence, all such x_j may be set equal to zero and dropped from further consideration in Phase II. If we drop them, our attention is confined only to variables whose corresponding $\bar{d}_j = 0$. By (9) solutions involving only these variables now have $w = 0$, and consequently are feasible for the original problem. Thus,

THEOREM 6: *If artificial variables form part of the basic sets of variables in the various cycles of Phase II, their values will never exceed zero.*

As one alternative to dropping variables x_j corresponding to $\bar{d}_j > 0$ at the end of Phase I, we can also maintain the basic artificial variables at zero values during Phase II by first eliminating (if possible) all artificial variables still in the basic set. This is done by choosing a pivot in a row r corresponding to such an artificial variable and in any columns s such that $\bar{a}_{rs} \neq 0$. If all coefficients in such a row for $j = 1, \ldots, N$ are zero, the row is deleted because the corresponding equation in the original system is redundant (see § 8-1).

As a second alternative, keep the w-equation during Phase II, and treat the $(-w)$ variable as just another variable which is restricted to nonnegative values. The system is then augmented by introducing the z-equation after eliminating the basic variables from it. Since $w \geq 0$ is always true, the added condition $(-w) \geq 0$ implies $w = 0$ during Phase II.

The computational procedures of Phase I with artificial variables and the transition to Phase II are summarized in the flow diagram, Fig. 5-2-I.

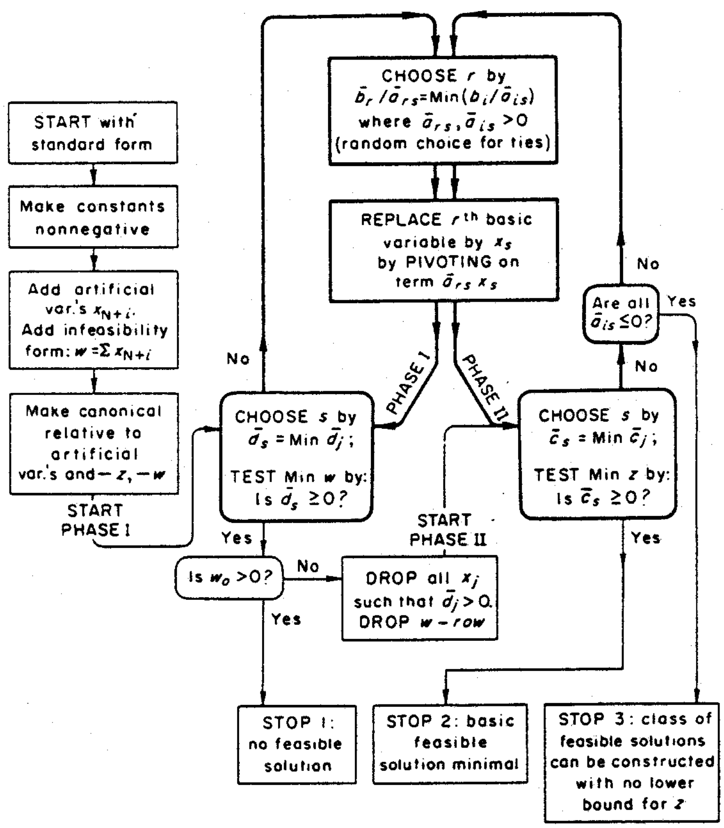

Figure 5-2-I. Flow diagram of the simplex method.

Detailed Iterative Procedure.

The *tableau* of the simplex method is given at various stages in Tables 5-2-I, II, and III. At the beginning of some cycle k all entries in a tableau associated with a cycle are known (see Table 5-2-II). Below each column corresponding to a basic variable, which includes $(-z)$ and $(-w)$, a ● symbol or ○ symbol is placed. Since the system is in canonical form (except that the original order of the variables has been preserved) all entries in the columns marked ● or ○ will be zero except one whose value is unity. If unity appears in the i^{th} row (except the last two), we will refer to the basic variable as the i^{th} basic variable and give it the symbol, x_{j_i}. For example, if unity occurs in the first, second, and third rows for basic variables x_3, x_5, and x_2, respectively, then $x_{j_1} = x_3$, $x_{j_2} = x_5$, and $x_{j_3} = x_2$ are the symbols entered in the left-hand margin of the tableau; their respective values in the corresponding basic solution are $\bar{b}_1, \ldots, \bar{b}_M$, which are shown in the last column, as are the values of the basic variables $(-z)$ and $(-w)$, which are the last two entries denoted by the symbols $-\bar{z}_0$ and $-\bar{w}_0$. The column of a variable entering the basic set on the next iteration is indicated by a ★; it replaces the basic variable indicated by a ○.

The following rules apply to all cycles but differ slightly depending on whether the computations are in Phase I or Phase II.

Step I:

(i) If all entries $\bar{d}_j \geq 0$ (in Phase I) or $\bar{c}_j \geq 0$ (in Phase II), then for

(a) Phase I with $\bar{w}_0 > 0$: *terminate*—no feasible solution exists.
(b) Phase I with $\bar{w}_0 = 0$: *initiate* Phase II by

(1) dropping all variables x_j with $\bar{d}_j > 0$,[4]
(2) dropping the w row of tableau, and
(3) restarting cycle (Step I) using Phase II rules.

(c) Phase II: *terminate*—an optimal solution is $x_{j_i} = \bar{b}_i$, $x_j = 0$, $z = \bar{z}_0$ $(j \neq j_i,\ i = 1, 2, \ldots, M)$.

(ii) If some entry $\bar{d}_j < 0$ (Phase I) or $\bar{c}_j < 0$ (Phase II), choose x_s as the variable to enter the basic set in the next cycle in place of the r^{th} basic variable (r to be determined in Step II), such that

$$\text{Phase I:} \quad \bar{d}_s = \operatorname{Min} \bar{d}_j < 0$$
$$\text{Phase II:} \quad \bar{c}_s = \operatorname{Min} \bar{c}_j < 0$$

[4] As an alternative, this step may be omitted and Step II-(ii) modified during Phase II as follows: If corresponding to an artificial basic variable x_{N+i} there is an $\bar{a}_{is} \neq 0$ for $s \leq N$, then drop the first such $i = r$ after pivoting on $\bar{a}_{rs}$; if none, perform Step II-(ii) as given.

TABLE 5-2-I

TABLEAU OF THE SIMPLEX METHOD

Initial Tableau, Cycle 0

Basic Variables	Admissible Variables	Artificial Variables				Objective Variables		Constants
	$x_1 \;\ldots\; x_s \;\ldots\; x_N$	x_{N+1}	x_{N+2}	$\ldots$	x_{N+M}	$-z$	$-w$	
x_{N+1}	$a_{11} \ldots a_{1s} \ldots a_{1N}$	1						b_1
x_{N+2}	$a_{21} \ldots a_{2s} \ldots a_{2N}$		1					b_2
$\vdots$	$\vdots \quad \vdots \quad \vdots$			$\ddots$				$\vdots$
x_{N+M}	$a_{M1} \ldots a_{Ms} \ldots a_{MN}$				1			b_M
$-z$	$c_1 \;\ldots\; c_s \;\ldots\; c_N$					1		0
$-w$	$-\sum a_{i1} \ldots -\sum a_{is} \ldots -\sum a_{iN}$						1	$-\sum b_i$
Basic Variables[1]	★	●	●		●	●	●	

← (these columns may be omitted)[2] →

TABLE 5-2-II

Tableau Start of Some Cycle k

x_{j_1}	$\bar{a}_{11}$		$\bar{a}_{1s}$		$\bar{a}_{1N}$	1					$\bar{b}_1$
x_{j_2}	$\bar{a}_{21}$		$\bar{a}_{2s}$		$\bar{a}_{2N}$		1				$\bar{b}_2$
$\vdots$	$\vdots$		$\vdots$		$\vdots$						$\vdots$
x_{j_r}	$\bar{a}_{r1}$		$\boldsymbol{\bar{a}_{rs}}$[3]	1	$\bar{a}_{rN}$						$\bar{b}_r$
$\vdots$	$\vdots$		$\vdots$		$\vdots$						$\vdots$
x_{j_M}	$\bar{a}_{M1}$	1	$\bar{a}_{Ms}$		$\bar{a}_{MN}$						$\bar{b}_M$
$-z$	$\bar{c}_1$		$\bar{c}_s$		$\bar{c}_N$				1		$-\bar{z}_0$
$-w$	$\bar{d}_1$		$\bar{d}_s$		$\bar{d}_N$					1	$-\bar{w}_0$
Basic Variables		●	★	○		●	●	(drop)	●	●	

[1] The ● or ○ indicates a column corresponding to a basic variable. All values in these columns are zero except one whose value is unity. The ★ indicates the position of most negative $\bar{d}_j < 0$, Phase I (or $\bar{c}_j < 0$, Phase II); i.e., the column of the variable entering the basic set on the next iteration by replacing the one indicated by ○.

[2] It is customary to omit the $-z$ and $-w$ columns because these remain the same through all tableaux and to omit the artificial variable columns because these, once dropped from the basic set, can be dropped from further consideration. Contrariwise, in the ***simplex method using multipliers*** (Chapter 9) the only entries recorded are those corresponding to the artificial variable columns.

[3] The bold-faced entry indicates position of pivot term for elimination for the next cycle; see Table 5-2-III.

TABLE 5-2-III

Tableau Beginning of Next Cycle, $k + 1$

Basic Variables	Admissible Variables			Artificial Variables		Objective Variables		Constants
	$x_1 \ldots$	$x_s \ldots$	x_N	x_{N+1}	$x_{N+2} \ldots x_{N+M}$	$-z$	$-w$	
x_{j_1}	$\bar{a}_{11} - \bar{a}_{1s}a^*_{r1}$		$\bar{a}_{1N} - \bar{a}_{1s}a^*_{rN}$	1				$\bar{b}_1 - \bar{a}_{1s}b^*_r$
x_{j_2}	$\bar{a}_{21} - \bar{a}_{2s}a^*_{r1}$		$\bar{a}_{2N} - \bar{a}_{2s}a^*_{rN}$		1			$\bar{b}_2 - \bar{a}_{2s}b^*_r$
.	.		.					.
x_s	a^*_{r1}	1	a^*_{rN}					b^*_r
.	.		.					.
x_{j_M}	$\bar{a}_{M1} - \bar{a}_{Ms}a^*_{r1}$	1	$\bar{a}_{MN} - \bar{a}_{Ms}a^*_{rN}$					$\bar{b}_M - \bar{a}_{Ms}b^*_r$
$-z$	$\bar{c}_1 - \bar{c}_s a^*_{r1}$		$\bar{c}_N - \bar{c}_s a^*_{rN}$			1		$-\bar{z}_0 - \bar{c}_s b^*_r$
$-w$	$\bar{d}_1 - \bar{d}_s a^*_{r1}$		$\bar{d}_N - \bar{d}_s a^*_{rN}$				1	$-\bar{w}_0 - \bar{d}_s b^*_r$
Basic Variables				●	●	(drop)		

where $(a^*_{r1} = \bar{a}_{r1}/\bar{a}_{rs}), \ldots, (a^*_{rN} = \bar{a}_{rN}/\bar{a}_{rs})$ $\qquad (b^*_r = \bar{b}_r/\bar{a}_{rs})$

Step II:

(i) If all entries $\bar{a}_{is} \leq 0$ *terminate*;[5] the class of solutions

$$x_s \geq 0 \text{ arbitrary}$$
$$x_{j_i} = \bar{b}_i - \bar{a}_{is}x_s \qquad (x_{j_i} \text{ basic variables})$$
$$x_j = 0 \qquad (x_j \text{ non-basic variables}; j \neq s)$$

satisfies the original system and has the property

$$z = \bar{z}_0 + \bar{c}_s x_s \to -\infty \text{ as } x_s \to +\infty$$

(ii) If some $\bar{a}_{is} > 0$, choose the r^{th} basic variable to drop in the next cycle, where

$$\bar{b}_r/\bar{a}_{rs} = \text{Min } \bar{b}_i/\bar{a}_{is}$$

and i and r are restricted to those i such that $\bar{a}_{is} > 0$. In case of ties[6] choose r at random (with equal probability) from those i which are tied.

Step III:

To obtain entries in the tableau for the next cycle from the current cycle, multiply each entry in the selected row r by the reciprocal of the pivot term $\bar{a}_{rs}$ and record the products in row r of the next cycle; see the starred entries in row r, Table 5-2-III. Enter the r^{th} basic

[5] In Phase I, this case cannot occur, for it would imply that w has no finite lower bound.

[6] See discussion on degeneracy, Chapter 10; see also § 6-1.

variable as x_s in place of x_{j_r} of the current cycle. To obtain the row i, column j entry of the next cycle, subtract from the corresponding entry of the current cycle the product of the entry in row i, column s of the current cycle and the entry in row r, column j of the next cycle.

Illustrative Example 1.

We shall now carry out the steps of the simplex method on our simple numerical example.

$$\begin{aligned} 5x_1 - 4x_2 + 13x_3 - 2x_4 + \ x_5 &= 20 \\ x_1 - \ x_2 + \ 5x_3 - \ x_4 + \ x_5 &= 8 \\ x_1 + 6x_2 - \ 7x_3 + \ x_4 + 5x_5 &= z \end{aligned}$$

Since the constant terms are nonnegative, we initiate Phase I of the simplex method with the augmented system

Admissible Variables	Artificial Variables		
$5x_1 - 4x_2 + 13x_3 - 2x_4 + x_5$	$+ x_6$		$= 20$
$x_1 - x_2 + 5x_3 - x_4 + x_5$	$+ x_7$		$= 8$
$x_1 + 6x_2 - 7x_3 + x_4 + 5x_5$		$-z$	$= 0$
	$x_6 + x_7$	$-w$	$= 0$

This is reduced to canonical form by subtracting the sum of the first two equations from the last. This then becomes the starting tableau for initiating Phase I. In order to show the relation between the ordinary elimination of a system of equations and the simplex algorithm, the computations are carried out in parallel in equation form in (10) and in tableau or detached coefficient form in Table 5-2-IV.

The steps for the minimization of w in Phase I are similar to those for minimizing z. The reader is referred to § 5-1, (4) through (14), for a detailed explanation for this example. On the first cycle the value of w is reduced from 28 to $\frac{4}{13}$, on the second cycle to zero, and a basic feasible solution $x_3 = \frac{3}{2}$, $x_5 = \frac{1}{2}$, $z = -8$ is obtained for the original unaugmented system. Variables x_6 and x_7 have positive relative cost factors for w and hence must be dropped for Phase II. On the third cycle the value of z dropped from $z_0 = -8$ (cycle 2) to $z_0 = -\frac{60}{7}$ which is minimum. The optimal solution is $x_2 = \frac{4}{7}$, $x_3 = \frac{12}{7}$, all other $x_j = 0$, $z = -\frac{60}{7}$.

(10) Simplex Method: Equation Form

Cycle 0 (Phase I)

$$\begin{aligned} 5x_1 - 4x_2 + \mathbf{13x_3} - 2x_4 + \ x_5 + x_6 \qquad\qquad &= \ 20 \\ x_1 - \ x_2 + \ 5x_3 - \ x_4 + \ x_5 \qquad + x_7 \qquad &= \ 8 \\ x_1 + 6x_2 - \ 7x_3 + \ x_4 + 5x_5 \qquad\qquad - z \quad &= \ 0 \\ -6x_1 + 5x_2 - 18x_3 + 3x_4 - 2x_5 \qquad\qquad - w &= -28 \end{aligned}$$

★ ○ ● ● ●

TABLE 5-2-IV

SIMPLEX METHOD: TABLEAU FORM

Cycle 0 (Phase I)

Basic Variables	Admissible Variables					Artificial Variables				Constants
	x_1	x_2	x_3	x_4	x_5	x_6	x_7	$-z$	$-w$	
x_6	5	-4	13	-2	$+1$	1				20
x_7	1	-1	$+5$	-1	$+1$		1			8
$-z$	1	6	-7	1	5			1		0
$-w$	-6	$+5$	-18	$+3$	-2				1	-28
			★			○	●	●	●	

Cycle 1 (Phase I)

x_3	$\frac{5}{13}$	$-\frac{4}{13}$	1	$-\frac{2}{13}$	$\frac{1}{13}$	$\frac{1}{13}$				$\frac{20}{13}$
x_7	$-\frac{12}{13}$	$+\frac{7}{13}$		$-\frac{3}{13}$	$\frac{8}{13}$	$-\frac{5}{13}$	1			$\frac{4}{13}$
$-z$	$\frac{48}{13}$	$\frac{50}{13}$		$-\frac{1}{13}$	$\frac{72}{13}$	$\frac{7}{13}$		1		$\frac{140}{13}$
$-w$	$+\frac{12}{13}$	$-\frac{7}{13}$		$+\frac{3}{13}$	$-\frac{8}{13}$	$\frac{18}{13}$			1	$-\frac{4}{13}$
			●		★	drop	○	●	●	

Cycle 2 (Phase I–II)

x_3	$\frac{1}{2}$	$-\frac{3}{8}$	1	$-\frac{1}{8}$		$+\frac{1}{8}$	$-\frac{1}{8}$			$\frac{3}{2}$
x_5	$-\frac{12}{8}$	$\frac{7}{8}$		$-\frac{3}{8}$	1	$-\frac{5}{8}$	$\frac{13}{8}$			$\frac{4}{8}$
$-z$	12	-1		2		4	-9	1		8
$-w$						1	1		1	0
		★	●		○	drop	drop	●	●	

Cycle 3 (Phase II–Optimal)

x_3	$-\frac{1}{7}$		1	$-\frac{2}{7}$	$\frac{3}{7}$	$-\frac{1}{7}$	$+\frac{4}{7}$		$\frac{12}{7}$
x_2	$-\frac{12}{7}$	1		$-\frac{3}{7}$	$\frac{8}{7}$	$-\frac{5}{7}$	$\frac{13}{7}$		$\frac{4}{7}$
$-z$	$\frac{72}{7}$			$\frac{11}{7}$	$\frac{8}{7}$	$\frac{23}{7}$	$-\frac{50}{7}$	1	$\frac{60}{7}$
		●	●			drop	drop	●	

(11) Cycle 1 (Phase I)

$$
\begin{array}{l}
\frac{5}{13}x_1 - \frac{4}{13}x_2 + x_3 - \frac{2}{13}x_4 + \frac{1}{13}x_5 + \frac{1}{13}x_6 \qquad\qquad = \frac{20}{13} \\
-\frac{12}{13}x_1 + \frac{7}{13}x_2 - \frac{3}{13}x_4 + \frac{6}{13}x_5 - \frac{5}{13}x_6 + x_7 \qquad = \frac{4}{13} \\
\frac{48}{13}x_1 + \frac{50}{13}x_2 - \frac{1}{13}x_4 + \frac{72}{13}x_5 + \frac{7}{13}x_6 \qquad - z \qquad = \frac{140}{13} \\
\frac{12}{13}x_1 - \frac{7}{13}x_2 + \frac{3}{13}x_4 - \frac{8}{13}x_5 + \frac{18}{13}x_6 \qquad\qquad - w = -\frac{4}{13}
\end{array}
$$

● ★ (drop) ○ ● ●

(12) Cycle 2 (Phase I–II)

$$
\begin{array}{l}
\frac{1}{2}x_1 - \frac{3}{8}x_2 + x_3 - \frac{1}{8}x_4 + \frac{1}{8}x_6 - \frac{1}{8}x_7 \qquad = \frac{3}{2} \\
-\frac{12}{8}x_1 + \frac{7}{8}x_2 - \frac{3}{8}x_4 + x_5 - \frac{5}{8}x_6 + \frac{13}{8}x_7 \qquad = \frac{4}{8} \\
12x_1 - x_2 + 2x_4 + 4x_6 - 9x_7 - z \qquad = 8 \\
x_6 + x_7 \qquad - w = 0
\end{array}
$$

★ ● ○ (drop) (drop) ● ●

(13) Cycle 3 (Phase II—Optimal)

$$
\begin{array}{l}
-\frac{1}{7}x_1 + x_3 - \frac{2}{7}x_4 + \frac{3}{7}x_5 - \frac{1}{7}x_6 + \frac{4}{7}x_7 \qquad = \frac{12}{7} \\
-\frac{12}{7}x_1 + x_2 - \frac{3}{7}x_4 + \frac{8}{7}x_5 - \frac{5}{7}x_6 + \frac{13}{7}x_7 \qquad = \frac{4}{7} \\
\frac{72}{7}x_1 + \frac{11}{7}x_4 + \frac{8}{7}x_5 + \frac{23}{7}x_6 - \frac{50}{7}x_7 - z = \frac{60}{7}
\end{array}
$$

● ● (drop) (drop) ●

Optimal Solution: $x_3 = \frac{12}{7}$, $x_2 = \frac{4}{7}$, all other $x_j = 0$, $z = -\frac{60}{7}$.

Illustrative Example 2.

TABLE 5-2-V

Simplex Method in Tableau Form for the Blending Problem, § 3-4

Cycle 0 (Phase I)

Basic Vari-ables	Admissible Variables									Artificial Variables						Con-stants
	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}	$-z$	$-w$	
x_{10}	1	1	1	1	1	1	1	1	1	1						[illegible]
x_{11}	.1	.1	.4	.6	.3	.3	.3	.5	.2		1					[illegible]3
x_{12}	.1	.3	.5	.3	.3	.4	.2	.4	.3			1				[illegible]3
x_{13}	+ .8	.6	.1	.1	.4	.3	.5	.1	.5				1			[illegible]4
$-z$	4.1	4.3	5.8	6.0	7.6	7.5	7.3	6.9	7.3					1		0
$-w$	−2.0	−2.0	−2.0	−2.0	−2.0	−2.0	−2.0	−2.0	−2.0						1	−2.0
	★									●	●	●	○	●	●	

Cycle 1 (Phase I)

Basic Vari-ables	Admissible Variables									Artificial Variables						Con-stants
	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}	$-z$	$-w$	
x_{10}		.250	.875	.875	.500	.625	.375	.875	.375	1						.50
x_{11}		.025	.388	.588	.250	.262	.238	.488	.138		1					.25
x_{12}		.225	+ .488	.288	.250	.362	.138	.388	.238			1				.25
x_1	1	.750	.125	.125	.500	.375	.625	.125	.625							[illegible]50
$-z$		1.22	5.29	5.49	5.55	5.96	4.74	6.39	4.74					1		−[illegible]05
$-w$		− .50	−1.75	−1.75	−1.00	−1.25	− .75	−1.75	− .75						1	−[illegible]00
	●		★							●	●	○	drop	●	●	

Cycle 2 (Phase I)

Basic Variables	Admissible Variables									Artificial Variables						Constants
	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}	$-z$	$-w$	
x_{10}		− .154		.359	.051	− .026	.128	.179	− .051	1						.05
x_{11}		− .154		+ .359	.051	− .026	.128	.179	− .051		1					.05
x_3		.462	1	.590	.513	.744	.282	.795	.487							.51
x_1	1	.692		.051	.436	.282	.590	.026	.564							.44
$-z$		−1.22		2.37	2.84	2.03	3.25	2.18	2.16					1		−4.76
$-w$		.31		− .72	− .10	.05	− .26	− .36	.10						1	− .10
	●		●	★						●	O	drop	drop	●	●	

Cycle 3 (Phase I–II)

Basic Variables	Admissible Variables									Artificial Variables						Constants
	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}	$-z$	$-w$	
x_{10}										1						0
x_4		− .428		1	.142	− .071	.357	.500	− .143							.14
x_3		+ .714	1		.428	.786	.071	.500	.571							.43
x_1	1	.714			.428	.286	.571	0	.571							.43
$-z$		− .20			2.50	2.20	2.40	1.00	2.50					1		−5.10
$-w$															1	0
	●	★	O	●						●	drop	drop	drop	●	●	

Drop w-equation after dropping all variables with $\bar{d}_j > 0$ (in this case w only).

Cycle 4 (Phase II—Optimal)

Basic Variables	Admissible Variables									Artificial Variables						Constants
	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}	$-z$	$-w$	
x_{10}										1						0
x_4			.6	1	.4	.4	.4	.8	.2							.4
x_2		1	1.4		.6	1.1	.1	.7	.8							.6
x_1	1															0
$-z$			.28		2.62	2.42	2.42	1.14	2.66					1		−4.98
	●	●								●	drop	drop	drop	●	drop	

5-3. PROBLEMS

1. What condition must be satisfied for a set of variables to be a basic set of variables? What is the difference between a feasible solution, a basic solution, a basic feasible solution, an optimal solution, and an optimal basic solution? Why is the term "an" optimal solution used instead of "the" optimal solution?

The Simplex Algorithm. (Refer to § 5-1 and § 5-2.)

2. Describe briefly in words the simplex algorithm. Make a "flow diagram" of the sequence of steps, cycles, etc. What is degeneracy?
3. Show for the redundant system

$$\begin{aligned} x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{22}x_2 + a_{33}x_3 &= 0 \\ -\, a_{22}x_2 - a_{33}x_3 &= 0 \end{aligned}$$

with $0 < a_{12} < 1$, $0 < a_{13} < 1$, $b_1 > 0$ that augmentation by artificial variables plus the usual Phase I procedure of the simplex method terminates with two artificial variables, and that the two equations associated with the artificial variables in the canonical form have one redundancy when the artificial variables are dropped but neither equation vanishes.

4. Show in general that if the original system is of rank r, i.e., has $m - r$ redundant equations, then there are at least $m' \geq m - r$ artificial variables left at the end of Phase I. If these artificial variables are dropped, then the subsystem of equations associated with these artificial variables is of rank $m' - (m - r)$, i.e., has also $m - r$ redundant equations. If $m' = m - r$, these equations are vacuous.
5. Discuss weaknesses and possible ways to improve the final solution to Phase I of the simplex method so as to have less Phase II cycles.
6. Show, by changing units of any activity k whose $\bar{c}_k < 0$, that it can be chosen by the rule of $\bar{c}_s = \text{Min } \bar{c}_j$ to be the candidate to enter the next basic set. Can you suggest another selection rule which might be better; does it involve more work?
7. What is a sufficient condition that an optimum solution be unique? If the condition is not satisfied, how can one go about constructing a different optimal solution if it exists?
8. Show that if $(x_1, x_2, \ldots, x_m)$ are basic variables, x_s can replace x_r as a basic variable only if the coefficient of $\bar{a}_{rs} \neq 0$ in canonical form.
9. Prove, using the method of artificial variables of Phase I of the simplex method, that if any feasible solution to a system in m linear equations in nonnegative variables exists, then one exists in which no more than m variables are positive.
10. (T. Robacker): In some applications it often happens that many variables initially in the basic set for some starting canonical form remain until the final canonical form, so that their corresponding rows in the successive tableaux of the simplex method, though continuously modified, have never been used for pivoting. Devise a technique for generating rows only as needed for pivoting and thereby avoiding needless work.
11. Suppose that in the canonical form at the end of Phase I with $w = 0$ an artificial variable remains in the basic set with its unit coefficient in row k. Show that any admissible variable x_j can replace the artificial one, providing $\bar{a}_{kj} \neq 0$. If all $\bar{a}_{kj} = 0$ for admissible j, the k^{th} row may be dropped from further consideration and this means that the k^{th} equation was redundant in the original system.
12. *Prove:* If there are no degenerate solutions after removal of the redundant equations, then the number of artificial variables at the end of Phase I, *without removal of these equations*, equals the number of redundant equations; and the equations, associated with the artificial variables in the canonical form (after dropping the artificial variables), are vacuous.

13. Identify the redundant equation if no artificial variable is allowed to re-enter when once dropped from a basic set. When can a class of solutions each having m variables with positive values (m = number of equations) have a lower bound of minus infinity?
14. Show that if the rank (see Problem 4) of a system of equations is the same as the number of equations and if feasible solutions exist, then basic feasible solutions exist; moreover if z has a finite lower bound a minimal basic feasible solution exists.
15. Discuss how the simplex method can be used to distinguish between a consistent system which is not solvable in nonnegative numbers and an inconsistent system.
16. How is redundancy identified in the simplex method?
17. Given a basic nonfeasible solution (i.e., at least one $\bar{b}_i < 0$) with all relative cost factors $\bar{c}_j \geq 0$, prove that $\bar{z}_0$ is a lower bound for possible values of z in § 5-1-(2).
18. Show that uniqueness of the canonical form means that there is one and only one linear form which can express a basic variable in terms of the non-basic variables. Use this to prove for the infeasibility form that the relative cost factors $\bar{d}_j = 0$ for non-artificial variables x_j, and $\bar{d}_j = 1$ for artificial variables, if the basic set of variables contains no artificial variables.
19. Show that the condition $\bar{c}_j \geq 0$ for all j is necessary for a nondegenerate basic feasible solution to be minimal.
20. Show that a degenerate basic feasible solution may be minimal without satisfying the condition $\bar{c}_j \geq 0$ for all j.
21. Show that no lower bound for z exists for the system

$$\begin{aligned} x_1 - x_2 &= 1 \qquad\qquad (x_1 \geq 0,\ x_2 \geq 0) \\ -x_1 - x_2 &= z \end{aligned}$$

and thus can be made to satisfy the conditions of § 5-1, Theorem 3.
22. In the following system one solution is $x_1 = 3$, $x_2 = 1$, $x_3 = 2$, $x_4 = 2$.

$$\begin{aligned} x_1 + x_2 - 2x_3 + x_4 &= 2 \qquad\qquad (x_j \geq 0) \\ x_1 - 2x_2 - x_3 + 2x_4 &= 3 \\ x_1 \qquad\qquad + 3x_4 &= 9 \end{aligned}$$

(a) Reduce to canonical form with respect to x_1, x_2, x_3; treat x_4 as an independent variable; and show how to reduce x_4 from its value $x_4 = 2$ toward zero and, at the same time adjust the values of the basic variables to obtain a solution with at most 3 variables positive.
(b) Find all solutions with at most 3 positive variables.

23. (a) Using the approach outlined in Problem 22 above, develop a variant of the simplex algorithm to reduce the number of positive variables

by at least one if the rank (see Problem 4) of their subsystem is less than their number. Under what circumstances can there be a change of more than one variable from a positive value to zero?

(b) Along the same lines as above, develop a variant of the simplex algorithm which begins with any feasible solution (basic or not) and by adjusting the values of non-basic variables up or down (if not at zero value), successively improves the solution towards optimality.

(c) Prove, using the above variant of the simplex algorithm, that (i) if feasible solutions exist then a basic feasible solution exists, (ii) if an optimal feasible solution exists then a basic feasible solution exists which is optimal, and (iii) if feasible solutions exist and the values of z associated with the solution set have a finite lower bound, then a basic feasible solution exists which is optimal.

24. If there is a feasible solution involving k variables, and if the rank (see Problem 4) of the subsystem formed by dropping the remaining variables is r, show that there is a feasible solution involving at most r variables where $r \leq k$.
25. If a system of m equations in n nonnegative variables has a feasible solution, then a solution exists in which k variables are positive and $n - k$ are zero, where $k \leq \text{Min}(m, n)$.
26. Show that in a nutrition problem with slacks where there is one food F that contains a little of each nutrient, there is a starting basic feasible solution involving $m - 1$ excess variables and the variable associated with F. Which excess variable is omitted?

The Two Phases of the Simplex Method. (Refer to § 5-2.)

27. Use the simplex method to solve the system

$$\begin{aligned} x_1 + x_2 &\geq 1 \\ x_1 + x_2 &\leq 2 \\ x_1 - x_2 &\leq 1 \\ x_1 - x_2 &\geq -1 \\ -x_2 &= z \end{aligned}$$

for nonnegative x_j and Min z. Plot the inequalities using x_1 and x_2 as coordinates, follow the solution steps graphically, and interpret the shift from one solution to the next on the graph. See Fig. 7-2-I.

28. [Waugh, 1951-1]: Dairy cows require a certain minimum combination of nutrients for maintenance and for milk production. Part of these requirements must be purchased. Given the following data, how much of each feed should the dairyman buy in order to supply all needed nutrients at the least possible cost? (Hint: Find proportions of requirements supplied by \$1 worth of each feed.)

A. Wholesale Prices and Nutritive Content of Feeds					
Feed	Wholesale Price, Kansas City, $/100 lbs.	Nutritive Content of Feeds (Pounds of each element in 100 pounds of feed)			
		Total Digestible Nutrients	Digestible Protein	Calcium	Phosphorus
Corn	2.40	78.6	6.5	0.02	0.27
Oats	2.52	70.1	9.4	0.09	0.34
Milo maize	2.18	80.1	8.8	0.03	0.30
Bran	2.14	67.2	13.7	0.14	1.29
Flour middlings	2.44	78.9	16.1	0.09	0.71
Linseed meal	3.82	77.0	30.4	0.41	0.86
Cottonseed meal	3.55	70.6	32.8	0.20	1.22
Soybean meal	3.70	78.5	37.1	0.26	0.59
Gluten feed	2.60	76.3	21.3	0.48	0.82
Hominy feed	2.54	84.5	8.0	0.22	0.71
B. Requirements for 24% total protein		74.2	19.9	0.21	0.67

29. Show that the feasible solution $x_1 = 1$, $x_2 = 0$, $x_3 = 1$, $z = 6$ to the system

$$\begin{aligned} x_1 + x_2 + x_3 &= 2 \qquad (x_j \geq 0)\\ x_1 - x_2 + x_3 &= 2\\ 2x_1 + 3x_2 + 4x_3 &= z \text{ (Min)} \end{aligned}$$

is not basic.

30. In the system below, the z form has all positive coefficients and $x_1 = x_2 = x_3 = x_4 = x_5 = 1$; $z = 5$ is a feasible solution. Without doing any calculations prove an optimal basic feasible solution must exist. Using Phase I and II of the simplex method construct an optimal solution.

$$\begin{aligned} z &= x_1 + x_2 + x_3 + x_4 + x_5 \qquad (x_j \geq 0, \text{ Min } z)\\ 2 &= 2x_1 + x_2 - x_3 + x_4 - x_5\\ 2 &= -x_1 + x_2 + 3x_3 - 2x_4 + x_5 \end{aligned}$$

31. Consider the system

$$\begin{aligned} 2x_1 - x_2 + x_3 &= 2 \qquad (x_1 \geq 0, x_2 \geq 0, x_3 \geq 0)\\ 4x_1 + x_2 + x_3 &= 6\\ x_1 + x_2 + x_3 &= z \end{aligned}$$

(a) What is the maximum number of solutions with at most two positive variables?

(b) Find all solutions with at most two positive variables. Which solution gives the smallest value of z?

(c) Reduce the problem to canonical form relative to x_1 and x_2. Is this solution optimal? If not, use the iterative procedure of the simplex algorithm to find the optimal solution. How does this agree with the result of (b)?

32. Find $x_j \geq 0$ and Min z for each of the following systems for the optimal solution:

(a)
$$\begin{aligned} 2x_1 - 3x_2 + x_3 + 3x_4 - x_5 &= 3 \\ x_1 + x_2 - 2x_3 + 9x_4 &= 4 \\ 2x_1 - 3x_2 + 6x_3 + x_4 - 2x_5 &= z \end{aligned}$$

(b)
$$\begin{aligned} 3x_1 + x_2 + 2x_3 + x_4 + x_5 &= 2 \\ 2x_1 - x_2 + x_3 + x_4 + 4x_5 &= 3 \\ x_1 - x_2 + 3x_3 - 2x_4 + x_5 &= z \end{aligned}$$

(c)
$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 2x_4 - x_5 &= 6 \\ 2x_2 + 4x_3 - 4x_4 + 2x_5 &= 6 \\ x_2 + x_3 + x_4 + x_5 &= 5 \\ -x_1 + 2x_2 + x_3 + 3x_4 - x_5 &= z \end{aligned}$$

33. Solve the Product Mix Problem of § 3-5 by the simplex method. Note that the model with the slack variables added is already in canonical form.

34. Using the simplex method, solve Problem 12, Chapter 3.

35. Solve the following problems by the simplex method. Verify your answers graphically (except c). Find $x_j \geq 0$, Min z satisfying

(a)
$$\begin{aligned} x_1 + x_2 - x_3 &= 2 \\ x_1 + x_2 + x_4 &= 4 \\ -2x_1 - x_2 &= z \end{aligned}$$

(b)
$$\begin{aligned} x_1 + x_2 + x_3 &= 2 \\ x_1 - 3x_2 - x_4 &= 3 \\ -2x_1 - x_2 &= z \end{aligned}$$

(c)
$$\begin{aligned} 2x_1 + x_2 - x_3 + x_4 &= 2 \\ 2x_1 - x_2 + 5x_3 + x_5 &= 6 \\ 4x_1 + x_2 + x_3 + x_6 &= 6 \\ -x_1 - 2x_2 - x_3 &= z \end{aligned}$$

(d)
$$\begin{aligned} -4x_1 + x_2 + x_3 &= 4 \\ 2x_1 - 3x_2 + x_4 &= 6 \\ -x_1 - 2x_2 &= z \end{aligned}$$

36. Is the solution of the Illustrative Example 1, § 5-2, unique? Give a rule for determining whether or not a solution is unique.

37. Solve for the optimal solution of each part of Problem 32, using artificial variables.

38. The problem of minimizing $4x_1 + 8x_2 + 3x_3$, subject to the five constraints

$$x_1 + x_2 \geq 2$$
$$2x_2 + x_3 \geq 5$$
$$x_j \geq 0 \qquad (j = 1, 2, 3)$$

may be converted into the following form, for immediate application of the simplex procedure:

Minimize $4x_1 + 8x_2 + 3x_3 + Wx_6 + Wx_7$, subject to the nine constraints:

$$\begin{aligned} x_1 + x_2 \quad - x_4 \quad + x_6 \quad &= 2 \\ 2x_2 + x_3 \quad - x_5 \quad + x_7 &= 5 \\ x_j &\geq 0 \qquad (j = 1, 2, \ldots, 7) \end{aligned}$$

where W is an arbitrarily large positive quantity.

(a) Explain the roles played by x_4 and x_5.
(b) Explain the roles played by x_6 and x_7.
(c) Why is it necessary to introduce x_6 and x_7, if x_4 and x_5 have already been introduced?
(d) What is the role played by W? Show that if W is large enough the sequence of steps is identical with the Phase I, Phase II procedure.
(e) Solve using the simplex method.

39. Minimize $-2y_1 - 5y_2$

subject to

$$\begin{aligned} y_1 \quad + y_3 \quad &= 4 \\ y_1 + 2y_2 \quad + y_4 \quad &= 8 \\ y_2 \quad + y_5 &= 3 \\ \text{and} \quad y_j &\geq 0 \end{aligned}$$

40. State and give the solution to the problem that is *dual* to the following problem.

Maximize $u_1 + u_2 + v_1 + v_2$

subject to $u_i + v_j = ij$ (the *product* of i and j; $i, j = 1, 2$)

A Nutrition Problem.

41. Formulate as a linear programming problem: Suppose six foods listed below have calories, amounts of protein, calcium, vitamin A, and costs per pound purchased as shown. In what amounts should these foods be purchased in order to meet exactly the daily equivalent per person shown in the last column at minimum cost? How is the model

modified if the daily requirements may be exceeded; if the requirements except for calories may be exceeded?

	Contents and Costs Per Pound Purchased						Daily Requirement
	Bread	Meat	Potatoes	Cabbage	Milk	Gelatin	
Calories	1254	1457	318	46	309	1725	3000
Protein	39	73	8	4	16	43	70 (grams)
Calcium	418	41	42	141	536	—	800 (mg.)
Vitamin A	—	—	70	860	720	—	500 (I.U.)
Cost	$0.30	$1.00	$0.05	$0.08	$0.23	$0.48	Minimum

(a) Reformulate the model with exact requirements if the unit of each activity is changed from a per pound purchased to a per 3,000 calories of bread, of meat, etc. purchased. Obtain graphically an optimal solution for a simplified problem in which the material balance equations for calories, proteins, and costs only are considered (i.e., those for calcium and vitamin A are dropped). Solve the full problem using the simplex method.

42. [Greene, Chatto, Hicks, and Cox, 1959-1]: Find the optimum plan for a meat packing plant that wishes to know what proportion of hams, bellies, and picnic hams should be processed for sale as smoked product, and what proportion should be sold fresh, or "green."

Maximum flow in the processing operation before overtime work is necessary on any given day is smoked ham = 106 (per 100 weight), total bellies and picnics = 315.

Total Amount of Fresh Product Available for Processing

Hams	Bellies	Picnics
480	400	230

Processing Costs in Dollars for Final Product

	Hams	Bellies	Picnics
Smoked product (Reg. time)	$5.18	$4.76	$5.62
Smoked product (Overtime)	$6.58	$5.54	$6.92
Green product	$ .50	$ .48	$ .51

Smoked products sell higher than green products: the difference between the selling prices for smoked and green hams = $6.00; between smoked and green bellies = $5.00; between smoked and green picnics = $6.00.

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CHAPTER 6

PROOF OF THE SIMPLEX ALGORITHM AND THE DUALITY THEOREM

6-1. INDUCTIVE PROOF OF THE SIMPLEX ALGORITHM

The proof given in § 5-1 of the simplex algorithm assumed all basic solutions generated by the iterative process to be nondegenerate. To cover the *degenerate* case there are two types of proofs available. The first, based on *induction*, has the advantage that at an early stage it yields a rigorous elementary proof of the fundamental duality theorem [Dantzig, 1959-1].

From a constructive viewpoint, the second proof, based on *perturbation* or *lexicographic* modification of the constant terms, has the advantage that it yields an easy rule for deciding which basic variable to drop when there is ambiguity [Dantzig, Orden, and Wolfe, 1954-1]. The proof requires, however, more background knowledge and is therefore postponed until Chapter 10. Either proof can be used to establish the simple *random choice rule* which requires the least work and guarantees with "probability one" that the simplex algorithm will terminate in a finite number of steps. Proof of the latter will be found at the end of this section. Our immediate objective is to show

THEOREM 1: *Given a linear program presented in feasible canonical form, there exists a finite sequence of pivot operations each yielding a basic feasible solution such that the final canonical form yields an optimal basic feasible solution, or an infinite class of feasible solutions for which the values of z have no lower bound.*

Discussion: For a linear program to be presented in feasible canonical form with the basic variables $x_1, \ldots, x_r, \ldots, x_m$, say, we must have

$$(1)\qquad \begin{array}{l} x_1 \qquad\qquad + \bar{a}_{1,m+1}x_{m+1} + \cdots + \bar{a}_{1s}x_s + \cdots + \bar{a}_{1n}x_n = \bar{b}_1 \\ \quad\vdots \qquad\qquad \cdots\cdots\cdots\cdots\cdots\cdots \qquad \vdots \\ \qquad x_r \qquad + \bar{a}_{r,m+1}x_{m+1} + \cdots + \bar{a}_{rs}x_s + \cdots + \bar{a}_{rn}x_n = \bar{b}_r \\ \qquad\quad\vdots \qquad \cdots\cdots\cdots\cdots\cdots\cdots \qquad \vdots \\ \qquad\qquad x_m + \bar{a}_{m,m+1}x_{m+1} + \cdots + \bar{a}_{ms}x_s + \cdots + \bar{a}_{mn}x_n = \bar{b}_m \\ \qquad\quad (-z) + \bar{c}_{m+1}x_{m+1} + \cdots + \bar{c}_s x_s + \cdots + \bar{c}_n x_n = -\bar{z}_0 \end{array}$$

where $\bar{z}_0$, $\bar{a}_{ij}$, and the $\bar{b}_i \geq 0$ are constants. (See § 5-1.) The basic feasible solution is obtained by assigning each of the non-basic variables the value zero and solving for the values of the basic variables, including z.

The simplex algorithm described in Chapter 5 may be outlined as follows: each iteration begins with a feasible canonical form with some set of basic variables. The associated basic solution is also *feasible*, i.e., the constants $\bar{b}_i$ (as modified) are nonnegative. The procedure terminates when a canonical form is achieved for which either $\bar{c}_j \geq 0$ for all j (in which case the basic feasible solution is optimal), *or* in some column with $\bar{c}_s < 0$, the coefficients are all nonpositive, $\bar{a}_{is} \leq 0$ (in which case a class of feasible solutions exists for which $z \to -\infty$). In all other cases a *pivot term* is selected in a column, s, and row, r, such that $\bar{c}_s = \text{Min } \bar{c}_j < 0$ and $\bar{b}_r/\bar{a}_{rs} = \text{Min } (\bar{b}_i/\bar{a}_{is})$ for $\bar{a}_{rs}$ and $\bar{a}_{is}$ positive. The variable x_s becomes a new basic variable replacing one in the basic set—namely, by using the equation with the pivot term to eliminate x_s from the other equations. When the coefficient of the pivot term is adjusted to be unity, the modified system is in canonical form, and a new basic feasible solution is available in which the value of $z = \bar{z}_0$ is decreased by a positive amount, if $\bar{b}_r > 0$. In the nondegenerate case, we have all $\bar{b}_i$'s positive. If this remains true from iteration to iteration, then a termination must be reached in a finite number of steps, because: (1) each canonical form is uniquely determined by choice of the m basic variables; (2) the decrease in value of $\bar{z}_0$ implies that all the basic sets are strictly different; (3) the number of basic sets is finite; indeed, not greater than the number of combinations of n things taken m at a time, $\binom{n}{m}$.

In the degenerate case it is possible that $\bar{b}_r = 0$; this results in $\bar{z}_0$ having the *same* value before and after pivoting. It has been shown by Hoffman and Beale (see § 10-1) that the procedure can repeat a canonical form and hence circle indefinitely. This phenomenon occurs, as can be inferred from what follows, when there is ambiguity in the choice of the pivot term by the above rules. A proper choice among them will always get around the difficulty. To show this we establish first the convenient lemma:

LEMMA 1: *If Theorem 1 holds for a system with at least one non-zero constant term, it holds for the system formed by replacing all constants by zero.*

PROOF: Suppose a system in canonical form has all constant terms zero. Change one or more $\bar{b}_i = 0$ to $\bar{b}_i' = 1$ (or any other positive value). Then, by hypothesis, there exists a sequence of basic feasible solutions obtained by pivoting, such that the final canonical form has the requisite properties. If exactly the same sequence of pivot choices are used for the totally degenerate problem, each basic solution remains feasible—namely zero. Since the desired property of the final canonical form depends only on the choice of basic variables, and not on the right-hand side, the lemma is demonstrated.

PROOF OF THEOREM 1. To establish the main theorem for the degenerate as well as the nondegenerate case we make the following

INDUCTIVE ASSUMPTION: *Assume for* $1, 2, \ldots, m-1$ *equations that only a finite number of feasible basic set changes are required to obtain a canonical form, such that the z-equation has all nonnegative coefficients* ($\bar{c}_j \geq 0$) *or some column s has* $\bar{c}_s < 0$ *and all nonpositive coefficients* ($\bar{a}_{is} \leq 0$).

We first verify the truth of the inductive assumption for one equation. If the initial basic solution is nondegenerate ($\bar{b}_1 > 0$), then we note that each subsequent basic solution must be nondegenerate (this remark holds only for the case of a single equation system). It follows that the finiteness proof of the simplex algorithm outlined above is valid, so that a final canonical form will be obtained that satisfies our inductive assumption. The degenerate case $\bar{b}_1 = 0$ is established by Lemma 1.

To establish the inductive step, suppose our inductive assumption holds for $1, 2, \ldots, m-1$ equations and that $\bar{b}_i \neq 0$ for at least one i in the m-equation system (1). If we are not at the point of termination, then the iterative process is applied until on some iteration a further decrease in the value of $\bar{z}_0$ is not possible, because of degeneracy. By rearrangement of equations, let $\bar{b}_1 = \bar{b}_2 = \ldots = \bar{b}_r = 0$ and $\bar{b}_i \neq 0$ for $i = r+1, \ldots, m$. Note that for *any* iteration, $r < m$ holds, because it is not possible to have *total* degeneracy on a subsequent cycle, if it is assumed that at least one of the $\bar{b}_i \neq 0$ initially. Let us set aside momentarily equations $r+1, \ldots, m$. According to our inductive assumption there exists a finite series of basic set changes, *using pivots from the first r equations*, that results in a subsystem satisfying all $\bar{c}_j \geq 0$, *or* for some s, all $\bar{a}_{is} \leq 0$, $1 \leq i \leq r$ and $\bar{c}_s < 0$. Let us perform these *same* pivots, but this time with the full system. *Since the constant terms for the first r equations are all zero, their values will all remain zero throughout the sequence of pivot term choices for the subsystem; this means we can apply the same sequence of choices for the entire system of m equations, without replacing* $x_{r+1}, \ldots, x_m$ *as basic variables or changing their values in the basic solutions.*

It follows then, that if the final basis for the subsystem has all $\bar{c}_j \geq 0$ then the same property holds for the system as a whole. If it has the property that for some s, $\bar{c}_s < 0$ and $\bar{a}_{is} \leq 0$ for $i = 1, 2, \ldots, r$, then either $\bar{a}_{is} \leq 0$ for all the remaining $i = r+1, \ldots, m$ (in which case the inductive property holds for m equations) or $\bar{a}_{is} > 0$ for at least one $i > r$, in which case the variable x_s can be introduced into the basic set for the system as a whole, producing a *positive decrease* in $\bar{z}_0$, since $\bar{b}_i > 0$ for $i = r+1, \ldots, m$. We have seen earlier that this value of z can decrease only a finite number of times. Hence, the iterative process must terminate, but the only way it can is when the inductive property holds for the m-equation system.

This completes the proof for m-equations, except for the completely degenerate case where $\bar{b}_i = 0$ for all $i = 1, 2, \ldots, m$. The latter proof, however, now follows directly from the lemma. Q.E.D.

As a corollary to Theorem 1 we have the following theorem.

THEOREM 2: *If there is only one choice of variable to drop under degeneracy, the simplex algorithm will terminate in a finite number of steps.*

Proof of the Random Choice Rule: This rule selects the variable to drop from the basic set with *equal probability* among those r, satisfying

$$\bar{b}_r/\bar{a}_{rs} = \text{Min } \bar{b}_i/\bar{a}_{is} \qquad (\bar{a}_{rs} > 0,\ \bar{a}_{is} > 0) \tag{2}$$

Starting with any basic feasible set, T, we know by Theorem 1, there exists a finite number of iterations leading to a final canonical form. Let k_T be the smallest number of iterations starting with T. Since there is only a finite number of starting basic sets, there exists a $k = \text{Max } k_T$, which is the *longest* of these shortest chains of steps.

LEMMA 2: *The random choice rule will terminate in k iterations with probability*

$$P \geq (1/m)^k \tag{3}$$

where m is the number of equations and k the longest of the shortest chain of steps leading to an optimal canonical form.

PROOF: There are m or less selections on each iteration. Thus, in k iterations, there are at most m^k sequences ("paths") of which at least one leads to an optimum; the probability of making a selection along such a path on each step is at least $(1/m)$, since we choose with equal probability. Hence for k steps (3) holds. Moreover, the probability of failing to reach an optimum before k iterations is less than $[1 - (1/m)^k]$. It follows that the probability of failing to reach an optimum by $2k$ iterations is less than $[1 - (1/m)^k]^2$ and failing to reach an optimum by $N = tk$ iterations is less than

$$[1 - (1/m)^k]^t \tag{4}$$

This expression, however, tends to zero as $t \to \infty$; therefore

THEOREM 3: *Given a random choice rule of which basic variable to drop from the basic set in case of a tie, the probability of failing to reach an optimum in N iterations tends to zero as $N \to \infty$.*

6-2. EQUIVALENT DUAL FORMS

As noted in § 3-8, associated with every linear programming problem is another linear programming problem called the *dual*. This fundamental notion was introduced by John von Neumann (in conversations with the author in October 1947) and appears implicitly in a working paper he wrote a few weeks later [von Neumann, 1947-1]. Subsequently Gale, Kuhn, and Tucker [1951-1] formulated an explicit Duality Theorem which they proved by means of the classical lemma of Farkas [1902-1]. Farkas's Lemma is described in § 6-4, Theorem 6. A systematic presentation of theoretical properties of dual linear programs will be found in Goldman and Tucker [1956-1], and Gale [1956-1]. A review of von Neumann's contributions can be found in Kuhn and Tucker [1958-1].

The original problem in its relation to the dual is called the *primal*. Feasible solutions to the primal and to the dual may appear to have little relation to one another; however, their optimum basic feasible solutions are such that it is possible to use one to obtain the other readily. It is often more convenient to use the dual to solve a linear programming problem than the primal. In this connection, it should be remarked that no advantage can be derived by solving the dual of the dual problem, because the latter turns out to be equivalent to the primal problem.

The Dual Problem.

A nearly symmetric relation between a primal problem and its dual problem results if the following system of linear inequalities (rather than equations) in nonnegative variables is considered.

Primal Problem: Find $x_j \geq 0$ and Min z, satisfying

$$\begin{array}{ll} & a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \geq b_1 \\ & a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \geq b_2 \\ (1) & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \geq b_m \\ & c_1x_1 + c_2x_2 + \ldots + c_nx_n = z \text{ (Min)}^1 \end{array}$$

In this form the dual problem is obtained by transposing the coefficient matrix, interchanging the role of the constant terms and the coefficients of the objective form, changing the direction of inequality, and maximizing instead of minimizing.

Dual Problem: Find $y_i \geq 0$ and Max v, satisfying

$$\begin{array}{ll} & a_{11}y_1 + a_{21}y_2 + \ldots + a_{m1}y_m \leq c_1 \\ & a_{12}y_1 + a_{22}y_2 + \ldots + a_{m2}y_m \leq c_2 \\ (2) & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & a_{1n}y_1 + a_{2n}y_2 + \ldots + a_{mn}y_m \leq c_n \\ & b_1y_1 + b_2y_2 + \ldots + b_my_m = v \text{ (Max)} \end{array}$$

This form of the dual problem, due to von Neumann, has the particular merit that it is easy to see that the dual of the dual is the primal (see Problem 5).

To see more clearly the connection between the primal and dual problems we shall use A. W. Tucker's detached coefficient array, Table 6-2-I. The primal problem reads across, the dual problem down. A simple way to remember the direction of inequality is to write the primal inequalities $\geq$

[1] "z (Min)" means z is to be minimized; not to be confused with Min z, which is the minimum value of z.

to correspond to the z-form, being always $\geq$ Min z, and to write the dual inequalities $\leq$ to correspond to the v-form, being always $\leq$ Max v.

TABLE 6-2-I

TUCKER DIAGRAM

	Primal						
	Variables	$x_1 \geq 0$	$x_2 \geq 0$	. . .	$x_n \geq 0$	Relation	Constants
Dual	$y_1 \geq 0$	a_{11}	a_{12}	. . .	a_{1n}	$\geq$	b_1
	$y_2 \geq 0$	a_{21}	a_{22}	. . .	a_{2n}	$\geq$	b_2
	.	.	.		.		.
	.	.	.		.		.
	.	.	.		.		.
	$y_m \geq 0$	a_{m1}	a_{m2}	. . .	a_{mn}	$\geq$	b_m
	Relation	$\leq$	$\leq$		$\leq$		$\leq$ Max v
	Constants	c_1	c_2	. . .	c_n	$\geq$ Min z	

The Duality Theorem is a statement about the range of possible z values for the primal versus the range of possible v values for the dual. This is depicted graphically in (3), for the case where the primal and dual are both feasible.

(3)

Dual		*Primal*	
$-\infty$ ——— v range ———→		←——— z range ——— $+\infty$	
or finite	Max v →	← Min z	or finite

DUALITY THEOREM. *If solutions to the primal and dual system exist, the value z of the objective form corresponding to any feasible solution of the primal is greater than or equal to the value v of the objective form corresponding to any feasible solution to the dual; moreover, optimal feasible solutions exist for both systems and* Max v = Min z.

The Dual of a Mixed System.

It is always possible to obtain the dual of a system consisting of a mixture of equations, inequalities (in either direction), nonnegative variables, or variables unrestricted in sign by reducing the system to an equivalent inequality system (1). In fact, this approach can be used to establish that the dual of a linear program in the standard form, as given in § 3-8, is the same as the one given here. Both the primal and dual systems can be viewed as consisting of a set of variables with their sign restrictions and a set of linear equations and inequalities, such that the variables of the primal are in one-to-one correspondence with the equations and inequalities of the dual, and the equations and inequalities of the primal are in one-to-one correspondence with the variables of the dual. When the primal relation is a linear inequality ($\geq$), the corresponding variable of the dual is nonnegative; if the relation is an equation, the corresponding variable will be unrestricted in sign. The following correspondence rules apply:

Primal	*Dual*
Objective Form ($\geq$ Min z)	Constant Terms
Constant Terms	Objective Form ($\leq$ Max v)
Coefficient Matrix	Transpose Coefficient Matrix
Relation:	Variable:
(i^{th}) Inequality: $\geq$	$y_i \geq 0$
(i^{th}) Equation: $=$	y_i unrestricted in sign
Variable:	Relation:
$x_j \geq 0$	(j^{th}) Inequality: $\leq$
x_j unrestricted in sign	(j^{th}) Equation: $=$

To illustrate, suppose we have the mixed primal system

(4)
$$\begin{aligned} x_1 - 3x_2 + 4x_3 &= 5 \qquad (x_1 \geq 0,\ x_2 \geq 0) \\ x_1 - 2x_2 \qquad &\leq 3 \qquad (x_3 \text{ unrestricted in sign}) \\ 2x_2 - \ \ x_3 &\geq 4 \\ x_1 + \ \ x_2 + \ \ x_3 &= z \text{ (Min)} \end{aligned}$$

Applying the rules, we have the primal system in detached coefficient form by reading across and the dual system reading down (Table 6-2-II).

TABLE 6-2-II

	Primal					
	Variables	$x_1 \geq 0$	$x_2 \geq 0$	x_3	Relation	Constants
Dual	y_1	1	-3	4	$=$	5
	$-y_2 \geq 0$	1	-2		$\leq$	3
	$y_3 \geq 0$		2	-1	$\geq$	4
	Relation	$\leq$	$\leq$	$=$		$\leq$ Max v
	Constants	1	1	1	$\geq$ Min z	

To see why this is the case, suppose we rewrite system (4) in equivalent inequality form (see § 4-5).

(5)
$$\begin{aligned} x_1 - 3x_2 + 4(x_3' - x_3'') &\geq 5, \qquad (x_1 \geq 0,\ x_2 \geq 0,\ x_3' \geq 0,\ x_3'' \geq 0) \\ -[x_1 - 3x_2 + 4(x_3' - x_3'')] &\geq -5 \\ -(x_1 - 2x_2) \qquad\qquad &\geq -3 \\ 2x_2 - \ \ (x_3' - x_3'') &\geq 4 \\ x_1 + \ \ x_2 + \ \ (x_3' - x_3'') &\geq \text{Min } z \end{aligned}$$

Here we have written $x_3 = x_3' - x_3''$ as *the difference of two nonnegative* *variables* and we have written the first equation of (4) as *equivalent to two inequalities*, $x_1 - 3x_2 + 4x_3 \geq 5$ and $x_1 - 3x_2 + 4x_3 \leq 5$. The relationship between the primal and dual by (1) and (2) is shown in Table 6-2-III.

TABLE 6-2-III

	Primal						
Dual	Variables	$x_1 \geq 0$	$x_2 \geq 0$	$x_3' \geq 0$	$x_3'' \geq 0$	Relation	Constants
	$y_1' \geq 0$	1	−3	4	−4	≥	5
	$y_1'' \geq 0$	−1	+3	−4	+4	≥	−5
	$y_2 \geq 0$	−1	+2			≥	−3
	$y_3 \geq 0$		2	−1	+1	≥	4
	Relation	≤	≤	≤	≤		≤ Max v
	Constants	1	1	1	−1	≥ Min z	

Here it is convenient to let $y_1' \geq 0$ and $y_1'' \geq 0$ be the dual variables corresponding to the first two inequalities. Since coefficients of y_1' and y_1'' differ only in sign in every inequality, we may set $y_1' - y_1'' = y_1$, where y_1 can have either sign. Note next that the coefficients in the inequalities of the dual corresponding to x_3' and x_3'' differ only in sign, which implies the equation

$$4(y_1' - y_1'') - y_3 = 1 \quad \text{or} \quad 4y_1 - y_3 = 1$$

From these observations it is clear that Table 6-2-II is the same as Table 6-2-III.

The Dual of the Standard Form.

We may apply the rules above to obtain the dual of the standard form; see Table 6-2-IV. It will be convenient to denote the dual variables (which in this case are unrestricted in sign) by $+\pi_i$ (instead of y_i in (2), which were restricted in sign).

TABLE 6-2-IV

	Primal						
Dual	Variables	$x_1 \geq 0$	$x_2 \geq 0$	. . .	$x_N \geq 0$	Relations	Constants
	$+\pi_1$	a_{11}	a_{12}	. . .	a_{1N}	=	b_1
	$+\pi_2$	a_{21}	a_{22}	. . .	a_{2N}	=	b_2 (Dual obj.)
	.	. . .	. . .	. . .	. . .	.	.
	.					.	.
	.					.	.
	$+\pi_M$	a_{M1}	a_{M2}	. . .	a_{MN}	=	b_M
	Relations	≤	≤	. . .	≤		≤ Max v
	Constants	c_1	c_2 (Primal objective)	. . .	c_N	≥ Min z	

6-3. PROOF OF THE DUALITY THEOREM

The primal problem for the standard linear program given in Table 6-2-IV is to choose variables $x_j \geq 0$ and Min z, satisfying

(1)
$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1N}x_N &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2N}x_N &= b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{M1}x_1 + a_{M2}x_2 + \ldots + a_{MN}x_M &= b_M \\ c_1x_1 + \quad c_2x_2 + \ldots + \quad c_Nx_N &= z \text{ (Min)} \end{aligned}$$

The dual problem for the standard linear program is to choose variables $\pi_1, \pi_2, \ldots, \pi_M$ and Max v, satisfying

(2)
$$\begin{aligned} a_{11}\pi_1 + a_{21}\pi_2 + \ldots + a_{M1}\pi_M &\leq c_1 \\ a_{12}\pi_1 + a_{22}\pi_2 + \ldots + a_{M2}\pi_M &\leq c_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{1N}\pi_1 + a_{2N}\pi_2 + \ldots + a_{MN}\pi_M &\leq c_N \\ b_1\pi_1 + \quad b_2\pi_2 + \ldots + \quad b_M\pi_M &= v \text{ (Max)} \end{aligned}$$

where π_i is unrestricted in sign.

All four combinations of feasibility and infeasibility of the primal and dual systems are possible. The four cases may be summarized as follows:

	Primal has feasible solution(s)	Primal has no feasible solution
Dual has feasible solution(s)	Min z = Max v	Max $v \to +\infty$
Dual has no feasible solution	Min $z \to -\infty$	Possible

The following examples show that each case is possible.

	Primal	Dual
(a) Primal feasible, Dual feasible	$x_1 \geq 0$ $x_1 = 5$ $x_1 = z$ (Min)	$\pi_1 \leq 1$ $5\pi_1 = v$ (Max)
	[Min z = Max v = 5]	
(b) Primal feasible, Dual infeasible	$x_1 \geq 0, x_2 \geq 0$ $x_1 - x_2 = 5$ $-x_1 - x_2 = z$ (Min)	$\pi_1 \leq -1$ $-\pi_1 \leq -1$ $5\pi_1 = v$ (Max)
	[$z \to -\infty$]	
(c) Primal infeasible, Dual feasible	$x_1 \geq 0$ $x_1 = -5$ $x_1 = z$ (Min)	$\pi_1 \leq 1$ $-5\pi_1 = v$ (Max)
	[$v \to +\infty$]	
(d) Primal infeasible, Dual infeasible	$x_1 \geq 0, x_2 \geq 0$ $x_1 - x_2 = +5$ $x_1 - x_2 = -5$ $-x_1 - x_2 = z$ (Min)	$\pi_1 + \pi_2 \leq -1$ $-\pi_1 - \pi_2 \leq -1$ $5\pi_1 - 5\pi_2 = +v$ (Max)

Proof of Duality Theorem and Related Theorems.

We shall use the simplex method to establish a group of fundamental theorems concerned with duality.

THEOREM 1: *Duality Theorem. If feasible solutions to both the primal and dual systems exist, there exists an optimum solution to both systems and*

$$\text{Min } z = \text{Max } v$$

THEOREM 2: *Unboundedness Theorem.*

(a) *If a feasible solution to the primal system exists, but not to the dual, there exists a class of solutions to the primal, such that $z \to -\infty$.*

(b) *If a feasible solution to the dual system exists, but not to the primal, there exists a class of solutions to the dual, such that $v \to +\infty$.*

THEOREM 3: *Infeasibility Theorem.*

(a) *If a system of linear equations in nonnegative variables is infeasible, there exists a linear combination of the equations which is an infeasible equation.*

(b) *If a system of linear inequalities is infeasible, there exists a nonnegative linear combination of the inequalities which is an infeasible inequality.*

Since a system of equations in nonnegative variables is equivalent to a linear inequality system, and conversely, Theorem 3(b) is a restatement of Theorem 3(a) in the equivalent system. Since the dual of a dual system is equivalent to the primal system, as we have just seen, Theorem 2(b) is a restatement of Theorem 2(a) for the dual system.

We shall, however, give direct proofs of all parts of these theorems by applying the simplex method. Before doing so, let us make a few preliminary observations that are related to the proof of the duality theorem.

When feasible solutions exist for both the primal and the dual problems, an important relation exists between the values of v and those of z, namely, the values of v are always less than (or equal to) the values of z. This was depicted in § 6-2-(3). To prove this, let ($x_1, x_2, \ldots, x_N$, and z) be any solution to the primal system (1), and let ($\pi_1, \pi_2, \ldots, \pi_M$, and v) be any solution to the dual system (2). Let us denote by $\bar{c}_j \geq 0$ the differences between the right and left members of (2), thus

$$c_j - \sum_{i=1}^{M} a_{ij}\pi_i = \bar{c}_j \qquad (j = 1, 2, \ldots, N) \tag{3}$$

If we multiply the first equation of the primal system (1) by π_1, the second by $\pi_2, \ldots$, and subtract the sum of the resulting equations from the z-equation, we obtain immediately

$$\bar{c}_1 x_1 + \bar{c}_2 x_2 + \ldots + \bar{c}_N x_N = z - v \tag{4}$$

The fact that $\bar{c}_j \geq 0$, $x_j \geq 0$ implies that all terms which appear on the left are nonnegative; hence, for any solution of the dual, $0 \leq z - v$ or,

$$z \geq v \tag{5}$$

Thus, when solutions to both the primal and dual systems exist, the value of z for any primal solution forms an *upper bound* for values of v, and the value of v of any dual solution forms a *lower bound* for values of z; therefore, it is not possible in this case for either $z \to -\infty$ or $v \to +\infty$. Thus it is clear, *if optimum solutions exist*[2] to the primal and dual problems, then for such solutions

$$\text{Min } z \geq \text{Max } v \tag{6}$$

This is known as the *weak form* of the Duality Theorem.

To establish Theorem 1, we consider an auxiliary problem formed from (1) by first changing the signs of the terms of each equation i (if necessary), so that $b_i \geq 0$, and then introducing an "error" or "artificial" variable $x_{N+i} \geq 0$. Let us define variables $w \geq 0$ and $w' \geq 0$ by

$$w = \sum_1^M x_{N+i}; \qquad w + w' = W \tag{7}$$

where w measures the total sum of errors x_{N+i}. W is some known upper bound on the total error, and $w' \geq 0$ measures the slack between w and W. For example, an upper bound which could be used for w is $W = \sum_1^M b_i$, which corresponds to the initial basic solution of Phase I (see § 5-2).

Auxiliary Problem. Find $x_j \geq 0$, w', z such that $z = \text{Min } z$, given that $w' = \text{Max } w'$, which satisfy

$$\begin{array}{llll}
(8) & a_{11} x_1 + \cdots + a_{1N} x_N + x_{N+1} & & = b_1 \\
 & \cdots\cdots\cdots\cdots\cdots & \ddots & \vdots \\
 & a_{M1}x_1 + \cdots + a_{MN}x_N & \qquad\qquad x_{N+M} & = b_M \\
 & & x_{N+1} + \cdots + x_{N+M} + (w') & = W \\
 & c_1x_1 + \cdots + c_Nx_N & (-z) & = 0
\end{array}$$

It will be noted that (8) is just the standard form for Phase I of the simplex method, if w' is replaced by $W - w$. It will be in canonical form with respect to $x_{N+1}, \ldots, x_{N+M}, w', -z$ after elimination of these variables from the w'-form. We can now proceed to maximize w', which means we are minimizing $w = W - w'$. Since a lower bound to w exists (namely 0), there exists by Theorem 1 of § 6-1 an optimal canonical form at termination of this Phase I, such that all the coefficients and the constant in the w'-equation (9) are nonnegative.

$$\sum_{j=1}^{N+M} d_j x_j + w' = +\bar{w}'_0 \qquad (d_j \geq 0,\ \bar{w}'_0 \geq 0) \tag{9}$$

[2] Notice that at this point we do not know that a minimizing solution to the primal or a maximizing solution to the dual exists.

On the other hand, this equation was generated from the auxiliary system (8) by a sequence of pivot operations; this implies that there exists some linear combination of the equations $i = 1, 2, \ldots, M$ of (8) with weights $\sigma_1^o, \sigma_2^o, \ldots, \sigma_M^o$, which, added to the w'-equation of (8), yields (9). The weights $\sigma_i = \sigma_i^o$ therefore satisfy

$$\begin{aligned}
&\sum_{i=1}^{M} \sigma_i^o a_{ij} = \bar{d}_j \geq 0, && \text{for } j = 1, 2, \ldots, N, \\
(10) \qquad &\sigma_i^o + 1 = \bar{d}_{N+i} \geq 0, && \text{for } i = 1, 2, \ldots, M, \\
&\sum_{i=1}^{M} \sigma_i^o b_i + W = \bar{w}_0' \geq 0
\end{aligned}$$

Taking this *same* linear combination of equations of the primal system (1), and setting $\bar{w}_0 = W - \bar{w}_0'$, yields

$$(11) \qquad \sum_{j=1}^{N} \bar{d}_j x_j = -\bar{w}_0 \qquad (\bar{d}_j \geq 0,\ \bar{w}_0 \geq 0)$$

In particular, if feasible solutions to (1) exist, $\text{Min}\, w = \bar{w}_0 = 0$. On the other hand, if no feasible solution to the primal exists, so that $\bar{w}_0 > 0$, then (11) becomes an infeasible equation in nonnegative variables x_j; this establishes Theorem 3(a).

Let us now assume a solution $(\pi_1 = \pi_1^o, \ldots, \pi_M = \pi_M^o)$ to the dual exists, so that

$$\begin{aligned}
(12) \qquad &\sum_{i=1}^{M} \pi_i^o a_{ij} \leq c_j \qquad (j = 1, 2, \ldots, N) \\
&\sum_{i=1}^{M} \pi_i^o b_i = v^o
\end{aligned}$$

then the dual relations (12) are also satisfied by the class of solutions $\pi_1 = (\pi_1^o - \theta\sigma_1^o), \ldots, \pi_M = (\pi_M^o - \theta\sigma_M^o)$, $v = v^o + \theta\bar{w}_0$, for any $\theta > 0$ because, by (12) and (10),

$$\begin{aligned}
(13) \qquad &\sum_{i=1}^{M} (\pi_i^o - \theta\sigma_i^o) a_{ij} = \sum_{i=1}^{M} \pi_i^o a_{ij} - \theta\bar{d}_j \leq c_j \\
&\sum_{i=1}^{M} (\pi_i^o - \theta\sigma_i^o) b_i = v^o + \theta\bar{w}_0 = v
\end{aligned}$$

Let us assume, in addition, that the primal problem is infeasible, so that $\text{Min}\, w = \bar{w}_0 > 0$. Then this class of solutions to the dual has the property that $v = v^o + \theta\bar{w}_0 \to \infty$ as $\theta \to +\infty$, establishing Theorem 2(b).

Our objective now is to seek a solution to our system (8), that *minimizes*

z for some specified value of W, starting with the last achieved canonical form (end of Phase I). The value of W that we choose at this stage may be the one we used initially or any other $W \geq \text{Min } w$. For example, we might redefine W to be Min w, as is customary in the usual Phase II procedure, in which case the value of the constant $\bar{w}_0'$ in the canonical form at the end of Phase I becomes $\bar{w}_0' = 0$. Whatever the choice of $W \geq \text{Min } w$, we shall refer to this as *the Phase II problem*.

According to Theorem 1 of § 6-1, if we begin with this adjusted canonical form, there exists a final canonical form, after a sequence of pivot operations, that yields either a solution that minimizes z or a class of solutions for which $z \to -\infty$. Let us consider the latter first.

The case $z \to -\infty$, for the auxiliary problem can arise only if some column, $j = s$, in the final canonical form (obtained at the end of Phase II), consists of all $\bar{a}_{is} \leq 0$ and $\bar{c}_s < 0$. We now observe that if an artificial variable, x_{N+i}, is in the final basic set, the corresponding row coefficient $\bar{a}_{is} = 0$, because otherwise an increase of the variable $x_s \to +\infty$ would generate an allowable class of solutions, with values of $x_{N+i} \to +\infty$, contradicting our hypothesis that $w = \sum x_{N+i} \leq W$. For the same reason x_s cannot correspond to any artificial variable x_{N+k}; hence, $1 \leq s \leq N$. In the final canonical form, we now note that we can obviously form the coefficients in column s as a linear combination of the coefficients[3] in columns corresponding to the basic variables $x_{j_1}, x_{j_2}, \ldots, x_{j_M}$; $-w$, $-z$ with weights $+\bar{a}_{1s}, +\bar{a}_{2s}, \ldots, +\bar{a}_{Ms}$; $\bar{d}_s$, $\bar{c}_s$ (because the matrix of coefficients of these columns is all zero, except for ones down the diagonal). This same linear combination must hold not only for the corresponding columns of the auxiliary system (8) but also for those of the primal system (1) because the weights $\bar{a}_{is}$ corresponding to augmented columns of (8), if any, have all zero values.[4] This is displayed in (14) in conventional matrix notation as discussed later in Chapter 8.

$$(14) \qquad \begin{bmatrix} a_{1j_1} & a_{1j_2} & \cdots & a_{1j_M} & 0 & 0 \\ a_{2j_1} & a_{2j_2} & \cdots & a_{2j_M} & 0 & 0 \\ \cdot & \cdot & & \cdot & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot & \cdot \\ a_{Mj_1} & a_{Mj_2} & \cdots & a_{Mj_M} & 0 & 0 \\ 0 & 0 & \cdots & 0 & +1 & 0 \\ c_{j_1} & c_{j_2} & \cdots & c_{j_M} & 0 & +1 \end{bmatrix} \begin{bmatrix} \bar{a}_{1s} \\ \bar{a}_{2s} \\ \cdot \\ \cdot \\ \cdot \\ \bar{a}_{Ms} \\ \bar{d}_s \\ \bar{c}_s \end{bmatrix} = \begin{bmatrix} a_{1s} \\ a_{2s} \\ \cdot \\ \cdot \\ \cdot \\ a_{Ms} \\ 0 \\ c_s \end{bmatrix}$$

[3] By a linear combination of columns we mean a column of numbers formed by multiplying the corresponding entries in each column by weights associated with the column and summing the products. See Chapter 8 where such operations on column "vectors" are discussed.

[4] *Exercise:* Show that if a certain linear combination of the columns of a linear system vanishes before pivoting, it will vanish after pivoting, and conversely.

Exercise: When is it valid to form linear combinations of inequalities to form a new inequality?

As applied to the coefficients in the z-equation, this yields, in particular, the relation $+c_{j_1}\bar{a}_{1s} + c_{j_2}\bar{a}_{2s} + \cdots + c_{j_M}\bar{a}_{Ms} + \bar{c}_s = c_s$. Since the *columns* of the primal are in one-to-one correspondence with the *linear inequalities* of the dual system, this and the other relations state that if we multiply inequality j_1 of the dual system (2) by $-\bar{a}_{1s} \geq 0$, inequality j_2 by $-\bar{a}_{2s} \geq 0, \ldots$, inequality j_M by $-\bar{a}_{Ms} \geq 0$, and inequality $j = s$ by $+1$, and then sum, we will form the infeasible inequality

$$0 \cdot \pi_1 + 0 \cdot \pi_2 + \cdots + 0 \cdot \pi_M \leq \bar{c}_s \qquad (\bar{c}_s < 0) \tag{15}$$

This proves that *the dual system is infeasible if* $z \to -\infty$ *for the auxiliary problem.*

The case of z having a finite lower bound for the auxiliary problem arises only if a canonical form is obtained for (8) at the end of Phase II, such that the coefficients are nonnegative in the z-equation,

$$\sum_{j=1}^{N+M} \bar{c}_j^* x_j + \pi_w^* w' = z - \bar{z}_0 \qquad (\bar{c}_j^* \geq 0,\ \bar{z}_0 = \text{Min } z) \tag{16}$$

On the other hand, this equation can be formed directly from (8) by taking some linear combination of equations $i = 1, 2, \ldots, M$ with weights $-\pi_i^*$, the w-equation with weight $+\pi_w^*$, and the z-equation with weight 1. Since coefficients of x_j for $j = 1, 2, \ldots, N$ are all zero in the w-equation, we have constructed a feasible solution to dual $\pi_i = \pi_i^*$,

$$\begin{aligned} &\sum_{i=1}^{M} \pi_i^* a_{ij} \leq c_j \qquad (j = 1, 2, \ldots, N) \\ &\sum_{i=1}^{M} \pi_i^* b_i = \bar{z}_0 + \pi_w^* \text{W} = v^* \end{aligned} \tag{17}$$

This proves that *the dual system is feasible if z has a finite lower bound for any auxiliary problem whatever be the choice of* $\text{W} \geq 0$. Thus feasibility of the primal depends on the outcome of Phase I and feasibility of the dual on the outcome of Phase II (independent of the outcome of Phase I).

Assuming infeasibility of the dual system of inequalities, so that $z \to -\infty$ for any $\text{W} \geq 0$, then we have constructed the infeasible inequality (15). Theorem 2(b) is thus established. If the primal problem is also feasible and W was replaced at the beginning of Phase II by $\text{W} = 0$, then a class of primal *feasible* solutions has been constructed at the end of Phase II such that the values of z tend to $-\infty$. This establishes Theorem 2(a).

Assuming a feasible solution to the primal exists and W replaced by $\text{W} = 0$ for Phase II and assuming a feasible solution to the dual exists so that Phase II has a finite lower bound, then setting $\text{W} = 0$ in (17), we have shown the existence of feasible solutions to both systems such that

Min $z = z_0 = v^*$. But any z associated with a primal feasible solution is an upper bound for v, hence

$$\text{Max } v = \text{Min } z \tag{18}$$

establishing the Duality Theorem (Theorem 1).

6-4. BASIC THEOREMS ON DUALITY

Consider a system in standard form—we now state and prove the following related and important theorems.

THEOREM 1: *If $(x_1^*, \ldots, x_N^*, z^*)$ is a feasible solution to the primal and $(\pi_1^*, \ldots, \pi_M^*, v^*)$ is a feasible solution to the dual, satisfying for $j = 1, 2, \ldots, N$,*

$$\bar{c}_j^* = c_j - \sum_{i=1}^{M} \pi_i^* a_{ij} \geq 0, \quad \sum_{1}^{M} \pi_i^* b_i = v^* \tag{1}$$

a necessary and sufficient condition for optimality of both solutions is

$$\bar{c}_j^* = 0 \quad \textit{for} \quad x_j^* > 0 \tag{2}$$

THEOREM 2: *If a feasible solution exists for the primal, and z has a finite lower bound, an optimal feasible solution exists.*

THEOREM 3: *If an optimal feasible solution exists for the primal, there exists an optimal feasible solution to the dual.*

PROOF OF THEOREM 1: Let $x_j \geq 0$ be any feasible solution satisfying § 6-3-(1), and π_i be any multipliers, such that $\bar{c}_j \geq 0$ (see § 6-3-(3)). If the first equation of § 6-3-(1) is multiplied by π_1, the second by $\pi_2, \ldots$, etc., and the weighted sum of the first M equations is subtracted from the z-equation, there results

$$\bar{c}_1 x_1 + \bar{c}_2 x_2 + \ldots + \bar{c}_N x_N = z - v \tag{3}$$

Since $\bar{c}_j \geq 0$ and $x_j \geq 0$ by hypothesis, the left-hand side is nonnegative term by term, hence always

$$v = \sum_{i=1}^{M} \pi_i b_i \leq z \tag{4}$$

and v is a lower bound for values of z. By the hypothesis of Theorem 1, there is a particular feasible solution $x_j = x_j^* \geq 0$, $z = z^*$, and particular multipliers, $\pi_i = \pi_i^*$ and $\bar{c}_j^*$, such that $\bar{c}_j^* = 0$, if $x_j^* > 0$. Substituting these values in (3), the left-hand side vanishes term by term and $v^* = z^*$, and we conclude, by § 6-3-(6), that Max $v = v^* = z^* =$ Min z.

To show the necessity part of Theorem 1, we assume $v^* = z^*$. Substituting into (3) *all terms on the left must vanish*, which means $\bar{c}_j^* = 0$ for $x_j^* > 0$.

PROOF OF THEOREM 2: A proof of this theorem was given in § 6-2 and is an immediate consequence of applying the simplex algorithm to the auxiliary problem specified there. We have shown that in a finite number of cycles the process will terminate because (a) no feasible solution exists, (b) a class of feasible solutions has been constructed for which $z \to -\infty$, or (c) a basic optimal feasible solution $x_j = x_j^*$ has been obtained. Since cases (a) and (b) are ruled out by hypothesis, the theorem follows.

PROOF OF THEOREM 3: Referring again to the auxiliary problem of § 6-3-(8), the hypothesis of Theorem 3 satisfies the case of a feasible primal and finite minimum z. Hence there exist optimal multipliers for the dual, namely π_i^*, v^* specified in § 6-3-(17), (18).

Complementary Slackness in the Primal and Dual Systems.

When the primal and dual systems are expressed as systems of inequalities, Theorem 1 takes on a more symmetric form.

Let $x_j \geq 0$ be any feasible solution satisfying § 6-2-(1) and $y_i \geq 0$ be any feasible solution satisfying § 6-2-(2). We write the former in standard-equality form: Find $x_j \geq 0$, Min z, satisfying

$$\text{(5)}\quad \begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n - x_{n+1} \qquad\qquad\qquad = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \qquad - x_{n+2} \qquad\quad = b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \qquad\qquad - x_{n+m} = b_m \\ c_1x_1 + c_2x_2 + \cdots + c_nx_n \qquad\qquad\qquad\qquad = z \text{ (Min)} \end{array}$$

where $x_{n+i} \geq 0$ are variables that measure the extent of inequality, or *negative slack*, between the left- and right-hand sides of the i^{th} inequality.

It will be convenient also to let y_{m+j} measure the *positive slack* in the j^{th} inequality, $j = 1, 2, \ldots, n$, of the dual system. Thus § 6-2-(2) in standard-equality form becomes: find $y_i \geq 0$, Max v satisfying

$$\text{(6)}\quad \begin{array}{l} a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m + y_{m+1} \qquad\qquad\qquad = c_1 \\ a_{12}y_1 + a_{22}y_2 + \cdots + a_{m2}y_m \qquad + y_{m+2} \qquad\quad = c_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m \qquad\qquad + y_{m+n} = c_n \\ b_1y_1 + b_2y_2 + \cdots + b_my_m \qquad\qquad\qquad\qquad = v \text{ (Max)} \end{array}$$

Multiplying the i^{th} equation of (5) by y_i, $i = 1, 2, \ldots, m$, and subtracting their sum from the z-form yields

$$\text{(7)}\quad (c_1 - \sum_{i=1}^{m} a_{i1}y_i)x_1 + (c_2 - \sum_{i=1}^{m} a_{i2}y_i)x_2 + \cdots + (c_n - \sum_{i=1}^{m} a_{in}y_i)x_n$$
$$+ y_1x_{n+1} + y_2x_{n+2} + \cdots + y_mx_{n+m} = z - \sum_{i=1}^{m} y_ib_i$$

or, from the definitions of y_{m+j} and v given in (6) we have,

$$(8) \quad (y_{m+1}x_1 + y_{m+2}x_2 + \cdots + y_{m+n}x_n) + (y_1x_{n+1} + y_2x_{n+2} + \cdots + y_nx_{n+m}) = z - v$$

The left-hand side of (8) is nonnegative term by term, hence $0 \leq z - v$ or $v \leq z$.

Since we are assuming that primal and dual solutions exist, the hypothesis of the Duality Theorem is satisfied and there exist optimal feasible solutions to both systems. We shall now prove

THEOREM 4: *For optimal feasible solutions of the primal and dual systems, whenever slack occurs in the k^{th} relation of either system, the k^{th} variable of its dual vanishes; if the k^{th} variable is positive in either system, the k^{th} relation of its dual is equality.*

PROOF: Let $x_j = x_j^* \geq 0$ $(j = 1, 2, \ldots, n)$, $z = z^*$ and $y_i = y_i^* \geq 0$ $(i = 1, 2, \ldots, m)$, $v = v^*$ be the values associated with an optimal solution to the primal and the dual, and let $x_{n+i}^* \geq 0$ and $y_{m+j}^* \geq 0$ be the corresponding values of the slack variables obtained by substitution in (5) and (6); then $z^* - v^* = \text{Min}\, z - \text{Max}\, v = 0$ by the fundamental theorem, so that the right-hand side of (8) vanishes. However, as noted in the sequel to (8), each term on the left is nonnegative and hence must vanish term by term; i.e., $y_{m+j}^*x_j^* = 0$ and $y_i^*x_{n+i}^* = 0$. However, the term $y_{m+j}^*x_j^* = 0$ is the product of the slack in the j^{th} relation of the dual and its corresponding variable in the primal; the term $y_i^*x_{n+i}^*$ is the product of slack in the i^{th} relation of the primal and its corresponding dual variable. Hence, if $y_{m+j}^* > 0$, then $x_j^* = 0$; similarly, if $x_{n+i}^* > 0$, then $y_i^* = 0$. This is a restatement of Theorem 1 on the correspondence between an optimal solution of the primal system and the corresponding slack relations of an optimal solution of the dual system.

Homogeneous Systems.

There are several important duality-type theorems that predated the linear programming era [Tucker, 1956-1]. The earliest known result on feasibility is one concerning *homogeneous systems* (systems with constant terms all zero).

THEOREM 5: [Gordan, 1873-1] *Either a linear homogeneous system of equations possesses a nontrivial solution in nonnegative variables or there exists an equation, formed by taking some linear combination, that has all positive coefficients.*

PROOF: Let the homogeneous system for $i = 1, 2, \ldots, m - 1$ be

$$(9) \qquad \sum_{j=1}^{n} a_{ij}x_j = 0 \qquad (x_j \geq 0)$$

If such a system possesses a nontrivial solution (not all $x_j = 0$), a solution exists that also satisfies

$$(10) \qquad \sum_{j=1}^{n} x_j = 1$$

We shall treat (10) as the m^{th} equation of the system. According to the Infeasibility Theorem, § 6-3, Theorem 3(a), *either* there exists a feasible solution *or* there exist multipliers $(\pi_1, \pi_2, \ldots, \pi_{m-1}; \pi_m)$, such that the resulting linear combination is an infeasible equation in nonnegative variables;

$$\sum_{j=1}^{n} \bar{d}_j x_j = -\bar{w}_0 \qquad \text{where } \bar{d}_j \geq 0,\ \bar{w}_0 > 0 \tag{11}$$

It follows under the second alternative that $\pi_m = -\bar{w}_0 < 0$ and

$$\sum_{i=1}^{m} a_{ij}\pi_i = \bar{d}_j - \pi_0 > 0 \qquad (j = 1, 2, \ldots, n) \tag{12}$$

Hence, if multipliers $(\pi_1, \pi_2, \ldots, \pi_m)$ are used to form the linear combination of equations, the coefficients given by (11) of the resulting equation are all positive.

Exercise: Show the converse of Gordan's Theorem, namely, if there exists a linear combination whose coefficients are all positive, the homogeneous system in nonnegative variables possesses only a trivial solution.

Theorem 6. [Farkas' Lemma, 1902-1] *If a linear homogeneous inequality,*

$$\sum_{i=1}^{m} \pi_i b_i \leq 0 \tag{13}$$

holds for all sets of values of π_i *satisfying a system of homogeneous inequalities*

$$\sum_{i=1}^{m} a_{ij}\pi_i \leq 0 \qquad (j = 1, 2, \ldots, n) \tag{14}$$

then the inequality is a nonnegative linear combination of the inequalities of the system.

Proof: Assume there exists no nonnegative linear combination of (14) that yields (13). Then there exists no feasible solution to the system

$$\sum_{j=1}^{n} a_{ij}x_j = b_i \qquad (x_j \geq 0) \tag{15}$$

By Theorem 3(a) of § 6-3, there exist multipliers $\pi_i = \pi_i^0$, which, when applied to (15), yield an infeasible equation; the coefficients of this equation are

$$\sum_{i=1}^{m} a_{ij}\pi_i^0 \leq 0 \qquad (j = 1, 2, \ldots, n) \tag{16}$$

$$\sum_{i=1}^{m} b_i \pi_i^0 = \bar{w}_0 > 0$$

which contradicts (13).

Exercise: What is the analogue of this theorem for linear equation systems?

Theorem 7: [Stiemke, 1915-1] *Either a linear homogeneous system possesses a solution with all variables positive, or there exists a linear combination that has all nonnegative coefficients, one or more of which are positive.*

Proof: If the homogeneous system possesses a strictly positive solution, there exists a solution to the system

$$\sum_{j=1}^{n} a_{ij} x_j = 0 \qquad (i = 1, 2, \ldots, m) \tag{17}$$

$$x_j \geq 1 \qquad (j = 1, 2, \ldots, n)$$

Replacing $x_j \geq 1$ by $x_j = x_j' + 1$, where $x_j' \geq 0$, results in the system

$$\sum_{j=1}^{n} a_{ij} x_j' = -\sum_{j=1}^{n} a_{ij} \qquad (x_j' \geq 0) \tag{18}$$

By Theorem 3(a) of § 6-3, *either* this system possesses a feasible solution (which is the first alternative), *or* there exist multipliers $\pi_1, \pi_2, \ldots, \pi_m$, such that the resulting linear combination

$$\sum_{j=1}^{n} \bar{d}_j x_j = -\bar{w}_0 \qquad (\bar{d}_j \geq 0, +\bar{w}_0 \geq 0). \tag{19}$$

is an infeasible equation in nonnegative variables. In the latter case $\bar{w}_0 > 0$) It is also easy to see that $\sum_1^n \bar{d}_j = \bar{w}_0$, because the negative sum of the coefficients of each equation (18), from which it was derived, equals the corresponding constant term. It follows that at least one coefficient $\bar{d}_j$ of this equation must be positive (which is the second alternative).

Motzkin's Transposition Theorem [1936-1].

Consider the dual linear programs satisfying the Tucker Diagram (20).

			Primal		
		Variables	$x_1 \geq 0, \ldots, x_k \geq 0$	$x_{k+1} \geq 0, \ldots, x_n \geq 0$	Constants
(20)	Dual	u_1	$a_{11} \ldots a_{1k}$	$a_{1k+1} \ldots a_{1n}$	$= 0$
		u_2	$a_{21} \ldots a_{2k}$	$a_{2k+1} \ldots a_{2n}$	$= 0$
		$\vdots$	$\ldots\ldots\ldots$	$\ldots\ldots\ldots$	$\vdots$
		u_m	$a_{m1} \ldots a_{mk}$	$a_{mk+1} \ldots a_{mn}$	$= 0$
		Relations	$\leq \qquad \leq$	$\leq \qquad \leq$	
		Constants	$0 \qquad 0$	$0 \qquad 0$	

We assume all columns are non-vacuous. Consider *any arbitrary subset* of k columns; for example the first k columns shown in (20) to the left of the vertical dashed line.

THEOREM 8: *Either there exists a solution to the dual system, such that all inequalities corresponding to the subset hold strictly, or the primal system has a solution, such that at least one corresponding variable has positive value.*

PROOF: If there exists a solution to the primal system with the requisite property, then one exists such that

$$x_1 + x_2 + \ldots + x_k = 1 \tag{21}$$

where $j = 1, 2, \ldots, k$ is the assumed subset. The remainder of the proof parallels that of Theorem 5.

Theorem of Alternatives for Matrices [Ville, 1938-1].

Consider the dual homogeneous programs with vacuous objective forms,

$$\sum_{j=1}^{n} a_{ij}x_j \geq 0, \qquad x_j \geq 0 \qquad (i = 1, 2, \ldots, m) \tag{22}$$

and

$$\sum_{i=1}^{m} a_{ij}y_i \leq 0, \qquad y_i \geq 0 \qquad (j = 1, 2, \ldots, n) \tag{23}$$

and let either system be the primal and the other the dual.

THEOREM 9: *Either there exists a solution to the primal where all inequalities hold strictly or there exists a nontrivial solution to the dual.*

EXERCISE: Show that this theorem is a special case of the Transposition Theorem by introducing slack variables into the primal system.

EXERCISE: Given two solutions to a homogeneous system (22), show that the sum of their corresponding values is also a solution.

EXERCISE: Suppose there exists a solution to a homogeneous system of inequalities all satisfied with strict equalities. Show that there exists a solution if the zero constants are all replaced by ones.

Tucker's Complementary Slackness Theorem [1956-1].

A sharper form of the Theorem of Alternatives can be obtained by judicious application of the Transposition Theorem.

THEOREM 10: *There exist solutions to the homogeneous dual programs* (22) *and* (23) *such that every variable and its complementary slack have one positive and one zero value.*

PROOF: Augment the systems with slack variables as in (5) and (6). Partition the primal system so that the subset consists of the one slack variable, x_{n+p}. By Theorem 8, a solution can be obtained such that either $x_{n+p} > 0$ for the primal system or $y_p > 0$ for the dual. If a solution to the primal exists with $x_{n+p} > 0$, let $x_j = x_j^p$ for $j = 1, 2, \ldots, n, \ldots, n + m$

be this solution, and let $y_i = y_i^p = 0$ for $i = 1, 2, \ldots, m, \ldots, m + n$ be an associated (trivial) solution to the dual. On the other hand, if a solution to the dual exists with $y_p > 0$, let the values of y_i for this solution be $y_i = y_i^p$ and let $x_j = x_j^p = 0$ be the values of x_j for an associated (trivial) solution to the primal. If now we *add* the corresponding values x_j^p and y_i^p for different p, we will obtain a pair of "composite" solutions to the primal and dual systems with the property that every slack variable of the primal or its corresponding dual variable has a positive value.

If we interchange the role of the primal and dual systems, we can generate another pair of composite solutions with the property that every variable of the (original) primal or its corresponding dual slack has positive value. Let us now add these two pairs of composite solutions. This will yield solutions to the primal and dual systems with the property that at least one member of each complementary pair is positive. The proof of Theorem 10 is completed by proving the following:

Exercise: Referring to (8), show for the homogeneous case (all $b_i = 0$, $c_j = 0$) every solution to the primal and dual systems is optimal and the products of all complementary pairs vanish.

6-5. LAGRANGE MULTIPLIERS

There is another way in which the dual system might arise. In the calculus if we wish to minimize a function z of two variables

$$F(x_1, x_2) = z \tag{1}$$

subject to the relation

$$G(x_1, x_2) = 0 \tag{2}$$

between x_1 and x_2, the standard procedure is to find the *unrestricted* minimum of the function Z given by

$$Z = F(x_1, x_2) - \pi G(x_1, x_2) \tag{3}$$

where π is a parameter, called the Lagrange multiplier, whose value will be specified later. If the unrestricted minimum of Z for some fixed value $\pi = \pi^o$ happens to be at values $x_1 = x_1^o$, $x_2 = x_2^o$ that satisfy (2), then these clearly are the values that minimize (1) subject to (2), since $Z = z$ for all (x_1, x_2) satisfying (2). We assume that a value of π can be found for which this is the case, and that at an unrestricted minimum the partial derivatives of Z with respect to x_1 and x_2 exist and vanish. This yields two equations in two unknowns, x_1 and x_2, which can be solved for x_1 and x_2 in terms of π. The value of π is obtained by substituting the expressions of x_1 and x_2 into (2); in other words, the value of π is then adjusted so that the unrestricted minimizing solution satisfies (2).

For example, let us find the point (x_1, x_2) on the unit circle the sum of whose coordinates, z, is a minimum:

$$(4)\qquad \begin{aligned} x_1^2 + x_2^2 &= 1 \\ x_1 + x_2 &= z \end{aligned}$$

We consider the unrestricted minimum of the expression

$$(5)\qquad Z = (x_1 + x_2) - \pi(x_1^2 + x_2^2 - 1)$$

At an unrestricted minimum the partials of Z with respect to x_1 and x_2 vanish:

$$(6)\qquad \begin{aligned} \frac{\partial Z}{\partial x_1} &= 0\colon\quad 1 - 2x_1\pi = 0 \\ \frac{\partial Z}{\partial x_2} &= 0\colon\quad 1 - 2x_2\pi = 0 \end{aligned}$$

Whence the minimizing solution is $x_1 = \frac{1}{2}\pi$, $x_2 = \frac{1}{2}\pi$. We now determine π, so that the solution satisfies the equation of the circle; substituting,

$$(7)\qquad (\tfrac{1}{2}\pi)^2 + (\tfrac{1}{2}\pi)^2 = 1$$

or $\pi = \pm\sqrt{2}/2$, whence $(x_1 = 1/\sqrt{2},\ x_2 = 1/\sqrt{2})$ or $(x_1 = -1/\sqrt{2},\ x_2 = -1/\sqrt{2})$. The first solution *maximizes* the sum of the coordinates, and the second (the solution sought) *minimizes*.

The same procedure is followed in general if the problem is to find values that minimize $F(x_1, x_2, \ldots, x_n) = z$, subject to the k restrictions

$$(8)\qquad \begin{aligned} G_1(x_1, x_2, \ldots, x_n) &= 0 \\ G_2(x_1, x_2, \ldots, x_n) &= 0 \\ &\cdots\cdots \\ G_k(x_1, x_2, \ldots, x_n) &= 0 \end{aligned}$$

In this case the unrestricted minimum of the function

$$(9)\qquad Z = F(x_1, x_2, \ldots, x_n) - [\pi_1 G_1(x_1, x_2, \ldots, x_n) + \pi_2 G_2(x_1, x_2, \ldots, x_n) + \cdots + \pi_k G_k(x_1, x_2, \ldots, x_n)]$$

is found, where the π_i, Lagrange multipliers, are unspecified constants to be determined later. It is assumed that values of π_i can be found so that the unrestricted minimum solution satisfies the restrictions. The n equations resulting from the vanishing of the n partial derivatives of this expression at a minimum are solved for $x_1, x_2, \ldots, x_n$ in terms of $\pi_1, \pi_2, \ldots, \pi_k$. These values are substituted into the k expressions $G_i(x_1, x_2, \ldots, x_n) = 0$, and the resulting k equations in $\pi_1, \pi_2, \ldots, \pi_k$ are solved for $\pi_1, \pi_2, \ldots, \pi_k$.

For example, consider the *linear programming problem*

$$(10)\qquad \begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \qquad (x_1 \geq 0, x_2 \geq 0, x_3 \geq 0) \\ x_1 + x_2 + x_3 &= z \text{ (Min)} \end{aligned}$$

This is equivalent to the system in (real) variables x_1, x_2, x_3 and the squares of real variables u_1, u_2, u_3:

$$\begin{array}{llllll} & & & & & \text{Lagrange multipliers:} \\ (11) & x_1 - u_1^2 & & & = 0 & : \bar{c}_1 \\ & & x_2 - u_2^2 & & = 0 & : \bar{c}_2 \\ & & & x_3 - u_3^2 & = 0 & : \bar{c}_3 \\ & x_1 & + 2x_2 & + 3x_3 & = 6 & : \pi \\ & x_1 & + x_2 & + x_3 & = z\ (\text{Min}) & \end{array}$$

where the first three equations replace the nonnegative restrictions. We now find the unrestricted minimum of the expression

$$(12)\quad Z = (x_1 + x_2 + x_3) - \bar{c}_1(x_1 - u_1^2) - \bar{c}_2(x_2 - u_2^2) - \bar{c}_3(x_3 - u_3^2) - \pi(x_1 + 2x_2 + 3x_3 - 6)$$

or

$$(13)\quad Z = 6\pi + (1 - \pi - \bar{c}_1)x_1 + (1 - 2\pi - \bar{c}_2)x_2 + (1 - 3\pi - \bar{c}_3)x_3 + \bar{c}_1u_1^2 + \bar{c}_2u_2^2 + \bar{c}_3u_3^2$$

The vanishing of the six partial derivatives yields, on slight rearrangement,

$$(14)\quad \begin{cases} \bar{c}_1 = 1 - \pi, & \bar{c}_1u_1 = 0, \\ \bar{c}_2 = 1 - 2\pi, & \bar{c}_2u_2 = 0, \\ \bar{c}_3 = 1 - 3\pi, & \bar{c}_3u_3 = 0. \end{cases}$$

To these relations we may further add, if we like, conditions that guarantee the existence of a minimum,

$$(15)\quad \bar{c}_1 \geq 0,\ \bar{c}_2 \geq 0,\ \bar{c}_3 \geq 0$$

for the function Z obviously does not possess an unrestricted minimum, if the coefficient $\bar{c}_j$ of u_j^2 is negative in (13).

In this case, if we try to solve explicitly (14) and (15) for x_j and u_j in terms of Lagrange multipliers, a distressing thing happens—there are no x_j terms; moreover, for each j there are two possibilities—either $\bar{c}_j = 0$ or $u_j = 0$. Noting that $x_j = u_j^2$, we may replace the condition $\bar{c}_ju_j = 0$ by $\bar{c}_jx_j = 0$, so that either $\bar{c}_j = 0$ or $x_j = 0$. Since $j = 1, 2, 3$, there is a total of 2^3 different cases to be considered; in the general linear programming problem as we shall see, there are 2^n cases to be considered. In view of (15) we may rewrite the Lagrange multiplier conditions for a minimum, as finding x_i and $\bar{\imath}_j$, $\bar{c}_i = 1, 2, 3$ such that

$$(16)\ (a)\quad x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0 \text{ satisfying } x_1 + 2x_2 + 3x_3 = 6,$$

$$(b)\quad \bar{c}_1 \geq 0,\ \bar{c}_2 \geq 0,\ \bar{c}_3 \geq 0,\ \pi \text{ satisfying } \begin{cases} \bar{c}_1 = 1 - \pi \\ \bar{c}_2 = 1 - 2\pi \\ \bar{c}_3 = 1 - 3\pi \end{cases}$$

$$(c)\quad \bar{c}_1x_1 = 0,\ \bar{c}_2x_2 = 0,\ \bar{c}_3x_3 = 0.$$

For the *general* linear programming problem, to determine $x_j \geq 0$ and Min z satisfying

(17)
$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \\ c_1x_1 + c_2x_2 + \ldots + c_nx_n &= z \end{aligned}$$

we replace the nonnegative relations by

(18)
$$x_j - u_j^2 = 0 \qquad (j = 1, 2, \ldots, n)$$

and seek an unrestricted minimum of the form

(19)
$$Z = \sum_{j=1}^{n} c_jx_j - \left[\pi_1\left(\sum_{j=1}^{n} a_{1j}x_j - b_1\right) + \ldots + \pi_m\left(\sum_{j=1}^{n} a_{mj}x_j - b_m\right)\right] - [\bar{c}_1(x_1 - u_1^2) + \ldots + \bar{c}_n(x_n - u_n^2)]$$

or

(20)
$$Z = \left(\sum_{i=1}^{m} \pi_ib_i\right) + \left(c_1 - \sum_{i=1}^{m} \pi_ia_{i1} - \bar{c}_1\right)x_1 + \ldots + \left(c_n - \sum_{i=1}^{m} \pi_ia_{in} - \bar{c}_n\right)x_n + \bar{c}_1u_1^2 + \bar{c}_2u_2^2 + \ldots + \bar{c}_nu_n^2$$

The function Z does not possess an unrestricted minimum unless (a) the coefficients of x_j vanish and (b) the coefficients of u_j^2 are nonnegative; hence we can further require, if we like, that the multipliers π_i and $\bar{c}_j$ satisfy for $j = 1, 2, \ldots, n$,

(21)
$$\bar{c}_j = c_j - [\pi_1a_{1j} + \pi_2a_{2j} + \ldots + \pi_ma_{mj}] \geq 0$$

Moreover, at the unrestricted minimum the partial derivative with respect to u_j must also vanish, yielding

(22)
$$\bar{c}_ju_j = 0 \quad \text{or} \quad \bar{c}_jx_j = 0 \qquad (j = 1, 2, \ldots, n)$$

If for *fixed* $\pi_j = \pi_j^*$, there exists $\bar{c}_j$ satisfying (21) and u_j or $x_j = u_j^2$ satisfying (22), this will clearly yield $\sum_1^m \pi_i^*b_i$, in (20), hence the true (global) minimum of Z (ruling out the possibility of a local minimum; see Fig. 7-1-VII). Since $Z = z$ for any $x_j = u_j^2$ and z satisfying (17), we conclude

THEOREM 1: *If there exist multipliers* $(\pi_i = \pi_i^*)$ *and* $(\bar{c}_j = \bar{c}_j^*)$ *satisfying* (21), *and variables* $(x_j = x_j^* \geq 0$, *and* $z = z^*)$ *satisfying* (17), *such that all products* $\bar{c}_j^*x_j^* = 0$, *then* $(x_1^*, \ldots, x_n^*, z^*)$ *is a minimizing solution.*

Conclusion.

If the linear programming problem is attacked by the method of Lagrange multipliers, we find that the multipliers, if they exist, must satisfy a "dual" system—namely, the linear inequality system (21), and maximize $v = \sum \pi_i b_i$ when conditions (22) pertain (see § 6-4, Theorem 1). Also the multipliers $\bar{c}_j$ (or relative cost factors) have the property that $\bar{c}_j x_j = 0$ for $j = 1, 2, \ldots, n$. The latter leads to 2^n possible cases of either $\bar{c}_j = 0$ or $x_j = 0$. It is here that the Lagrange multipliers approach breaks down, for it is not practical to consider all the 2^n cases for large n.

In a certain sense the simplex method can be viewed as a systematic way to eliminate most of the cases and to consider only a few. Indeed, it immediately restricts the number of cases by considering only those with $n - m$ of the $x_j = 0$ at one time and such that the determinant of the remaining m variables is non-zero and the unique value of these variables is positive (under nondegeneracy). The conditions $\bar{c}_j x_j = 0$ tell us that $\bar{c}_j = 0$ for $x_j > 0$, and this determines uniquely π_i and the remaining $\bar{c}_j$. If not all $\bar{c}_j \geq 0$, the case is dropped and a special new one is examined on the next iteration, and so on.

6-6. PROBLEMS

1. Prove that the optimal dual solution is never unique if the optimal primal basic solution is degenerate and the optimal dual is not.
2. Show that if no artificial variables remain at the end of Phase I, $\sigma_i^0 = 0$ for $i = 1, 2, \ldots, M$. See § 6-3-(10).
3. Prove: If there exists one nondegenerate optimal basic feasible solution the optimal dual multipliers π_i are unique. (See § 6-3.)
4. Prove: All $\bar{d}_j = 0$ at end of Phase I if there are no artificial variables in the basic solution except w'. (See § 6-3-(11).)
5. Show that the dual of the dual is the primal by reversing first all signs in § 6-2-(2), to have it in standard inequality form for finding the dual.
6. Let the "dual" be alternatively defined by transposing and changing the sign of the coefficient matrix, including the interchange of (and change of sign of) the constant terms and coefficients of the objective form, maintaining the same direction of inequality, and minimizing. Show in this form that the proof of "the dual of the dual is the primal" is immediate and that this definition of the dual is equivalent to the one of § 6-2.
7. Show that neither the primal nor the dual of the system

$$\begin{aligned} x_1 - x_2 &\geq 2 \qquad\qquad (x_1 \geq 0,\ x_2 \geq 0) \\ -x_1 + x_2 &\geq -1 \\ x_1 - 2x_2 &= z \text{ (Min)} \end{aligned}$$

has a feasible solution.
8. Construct other examples to illustrate all four cases of primal and dual feasibility and infeasibility.
9. Is it possible for the primal and dual problems § 6-3-(1), (2) to have solutions if the restrictions $x_j \geq 0$, $y_i \geq 0$ are removed, but no solutions if the restrictions are included?

10. Prove in general that an equation in the primal corresponds to an unrestricted variable in the dual and a variable unrestricted in sign corresponds to an equation.
11. Suppose z^o; $x_1^o > 0$, $x_2^o > 0$, . . ., $x_k^o > 0$ and $x_{k+1}^o = \ldots = x_n^o = 0$ constitute a feasible solution to a linear program. Show that, if the canonical form for the subsystem formed by dropping $x_{k+1}, \ldots, x_n$ has less than k equations, a new solution can be formed involving fewer variables with positive values and a value of z not greater than z^o. Show that this process can be repeated until a subsystem is formed with an equal number of variables with positive values, as in its canonical form. Show that this solution is unique if all other variables are zero.
12. Apply the results of Problem 11 to give a direct proof that if a feasible solution to a linear program exists, and if the values of z have a finite lower bound, then an optimal feasible solution also exists.
13. Assuming Farkas' Lemma is true, derive the Duality Theorem.
14. (a) Consider the following "game" problem; find $y_j \geq 0$, Min M satisfying

$$\sum_{j=1}^{n} y_j = 1 \qquad (i = 1, 2, \ldots, m)$$

$$\sum_{j=1}^{n} a_{ij} y_j \leq \mathrm{M}$$

Show that the dual is to find $x_i \geq 0$ and Max N satisfying

$$\sum_{i=1}^{m} x_i = 1$$

$$\sum_{i=1}^{m} x_i a_{ij} \geq \mathrm{N}$$

(b) Prove $\mathrm{N} \leq \sum_{i=1}^{m} \sum_{j=1}^{n} x_i a_{ij}\, y_j \leq \mathrm{M}$ and Max N = Min M.

(c) Prove that feasible solutions to primal and dual systems always exist.

(d) Why is Max N = Min M positive, if all $a_{ij} > 0$? See Chapter 13 for application of this type of system.

15. Find the dual of a *bounded variable* linear program:

$$\alpha_j \leq x_j \leq \beta_j \qquad (j = 1, 2, \ldots, n)$$

$$\sum_{j=1}^{n} a_{ij} x_j = b_i \qquad (i = 1, 2, \ldots, m)$$

$$\sum_{j=1}^{n} c_j x_j = z \text{ (Min)}$$

16. The Fourier-Motzkin elimination method permits one to drop a variable by increasing the number of inequalities. Dualize the procedure and

find a method for decreasing the number of inequalities by increasing the number of variables.

17. Suppose that an optimal solution with respect to a given objective form z is not unique and that it is desired to introduce an alternative objective z^* and to minimize z^*, given that z is minimum. Show that an optimal solution exists which is basic in the restraint system, excluding the z and z^* forms. Prove that this solution can be obtained by first dropping all variables x_j, such that $\bar{c}_j > 0$ at the end of Phase II, and then replacing the z-form by the z^* form.
18. Generalize the usual Phase I, Phase II procedure to find a solution that is as "feasible as possible" (Min w) and given that it is and is not unique, find the one which minimizes z, given that $w = \text{Min } w$.
19. Show that it is not possible for $z \to -\infty$, if no positive combination of activities vanishes. Discuss what this means in a practical situation if a positive combination vanishes except for a positive cost, a negative cost, a zero cost. Show that if $z \to -\infty$, there exists a homogeneous feasible solution to the system. Show that it is possible to have $z \to +\infty$ and $z \to -\infty$ in the same system.
20. Generalize the Phase I procedure to allow an artificial variable to have either sign. Allow the variable entering to increase as long as the sum of the absolute values of the artificial variables decreases.
21. Prove that if an optimal solution $x_j^o \geq 0$, $z = z^o = \text{Min } z$ exists, then the system of equations formed by dropping all x_j, such that $x_j^o = 0$ and setting $z = z^o$, is redundant.
22. Does a column with all negative entries in the original tableau imply that (if feasible solutions exist) a class of solutions exists such that $z \to -\infty$?

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CHAPTER 7

THE GEOMETRY OF LINEAR PROGRAMS

7-1. CONVEX REGIONS

Convex Two-Dimensional Regions.

The set of points (x_1, x_2) satisfying the relation

$$x_1 + x_2 \geq 2 \quad (1)$$

consists of a region in two-dimensional space on one side of the line (see Fig. 7-1-Ia)

$$x_1 + x_2 = 2 \quad (2)$$

This is an example of a convex region, or, what is the same thing, a convex set of points. The region defined by the angle between two lines, Fig. 7-1-Ib, is also a convex set.

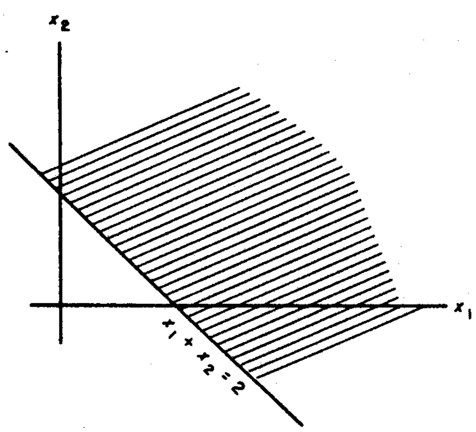


Figure 7-1-Ia.

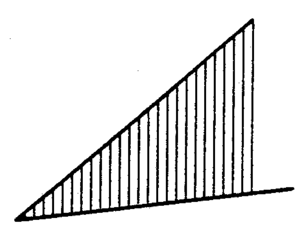


Figure 7-1-Ib.

Other examples in two dimensions are the region inside the rectangle, Fig. 7-1-IIa; the circle, Fig. 7-1-IIb; or the polygon, Fig. 7-1-IIc.

In three dimensions the volumes inside a cube and inside a sphere are also convex sets. The region defined may include or exclude the boundary. It may be bounded in extent or unbounded.

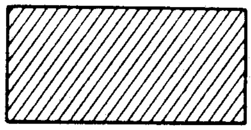


Figure 7-1-IIa.

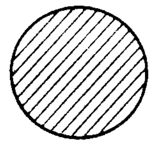


Figure 7-1-IIb.

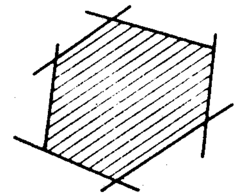


Figure 7-1-IIc.

On the other hand the sets depicted by the shaded region in Figs. 7-1-IIIa, IIIb, IIIc are not convex.

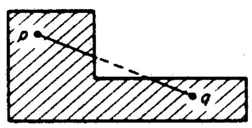


Figure 7-1-IIIa.

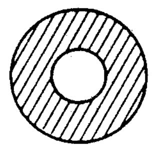


Figure 7-1-IIIb.

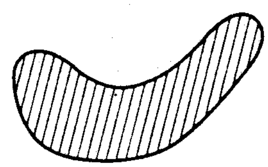


Figure 7-1-IIIc.

DEFINITION: A set of points is called a *convex* set if all points on the straight line segment joining any two points in the set belong to the set.

DEFINITION: A *closed* convex set is one which includes its boundaries. (For example, a circle and its interior is a closed convex set; the interior of a circle is a convex set, but is not closed.)

Thus the "L" shaped region of Fig. 7-1-IIIa is not a convex set because it is possible to find two points, p and q, in the set such that not all points on the line joining them belong to the set.

THEOREM 1: *The set of points common to two or more convex sets is convex.*

For example, the region common to two circles, Fig. 7-1-IVa, is convex or the points in the intersection of two or more regions defined by linear

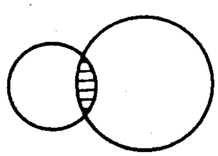


Figure 7-1-IVa.

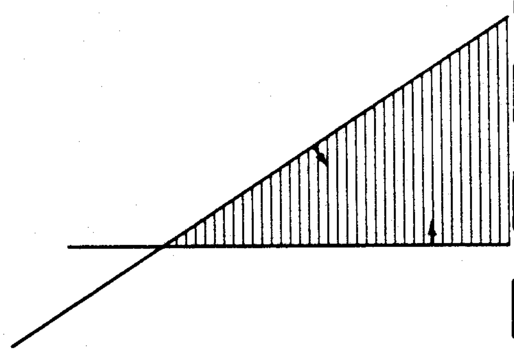


Figure 7-1-IVb.

inequalities form a convex region, Figs. 7-1-IVb, IVc and Figs. 7-1-Ib, IIc. In § 4-3, a succession of convex regions of feasible solutions was formed by successively adding restrictions (§ 4-3-(1), Fig. 4-3-I, and sequel).

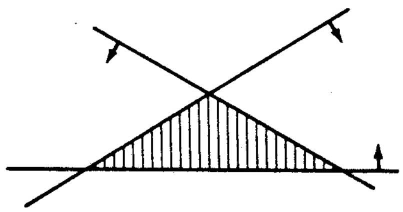


Figure 7-1-IVc.

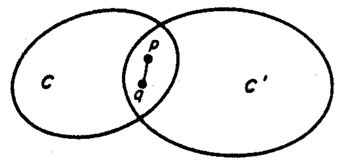


Figure 7-1-V.

Proof: Let C and C' be two convex sets and R the set of points common to C and C' (see Fig. 7-1-V). Let p and q be any two points in R. Since p and q are also in C and since C is convex, then the line segment joining p to q must be in C; for a similar reason the segment must be in C'. Hence the segment lying in both C and C' is in R.

Exercise: Extend the proof to more than two convex regions.

General Convex Regions.

Since in linear programming we will be dealing with linear inequalities involving many variables, it will not be possible to visualize the solution as a point in many dimensions. Accordingly we must be able to demonstrate algebraically whether or not certain sets are convex. The definition of a convex set requires that all points on a straight line segment joining any two points in the set belong to the set. It will be necessary to define in general what is meant by a "point" and a "straight line segment."

Definition: By a *point* x in n dimensions is meant an ordered set of n values or coordinates $(x_1, x_2, \ldots, x_n)$. The coordinates of x are also referred to as the *components* of x.

Definition: The *line segment* joining two points, p and q, with coordinates $(p_1, p_2, \ldots, p_n)$ and $(q_1, q_2, \ldots, q_n)$, respectively, in n-dimensional space is all points x whose coordinates are

$$(3) \qquad \begin{cases} x_1 = \lambda p_1 + (1 - \lambda)q_1 \\ x_2 = \lambda p_2 + (1 - \lambda)q_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ x_n = \lambda p_n + (1 - \lambda)q_n \end{cases}$$

where λ is a parameter such that $0 \leq \lambda \leq 1$. For example, consider the two points in two-dimensional space: $p = (6, 2)$ and $q = (1, 4)$. These are represented geometrically in Fig. 7-1-VI.

Consider now the point x, with coordinates (x_1, x_2). By definition if x is to be on the line segment joining p and q, then

$$(4) \qquad \begin{cases} x_1 = \lambda p_1 + (1 - \lambda)q_1 = 6\lambda + 1(1 - \lambda) = 5\lambda + 1 \\ x_2 = \lambda p_2 + (1 - \lambda)q_2 = 2\lambda + 4(1 - \lambda) = -2\lambda + 4\,. \end{cases}$$

For example, let $\lambda = 1$, then $x_1 = 6$ and $x_2 = 2$ and the point x is point p. Likewise let $\lambda = 0$, then $x = q$. For other λ values $(0 < \lambda < 1)$ we get all

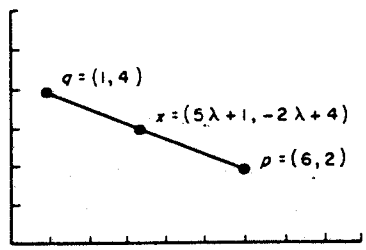


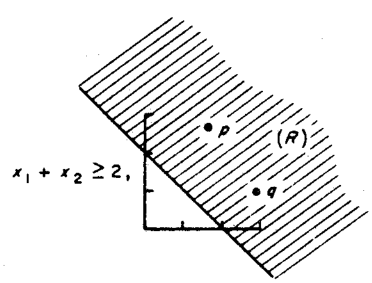
Figure 7-1-VI.

points between p and q. For example, when $\lambda = \frac{1}{2}$, the coordinates of x become $x_1 = \frac{7}{2}$, $x_2 = 3$ which is the point midway between p and q.

Exercise: Obtain the straight line relationship between x_1 and x_2 by eliminating λ in (4).

With this definition of a line segment, it is possible to determine whether a given set is convex. For example, consider the region R defined by all points whose coordinates satisfy

(5)



To prove that this region is convex, let $p = (p_1, p_2)$ and $q = (q_1, q_2)$ be any two points in R. For p and q to be in R their respective coordinates must satisfy (5), whence

$$(6) \qquad \begin{aligned} p_1 + p_2 &\geq 2 \\ q_1 + q_2 &\geq 2 \end{aligned}$$

Then the coordinates (x_1, x_2) of an arbitrary point, x, on the segment joining p to q, are found by forming a weighted combination of the coordinates of the two points as in (7) and (8).

$$
\begin{array}{ll}
 & \cdot\ p = (p_1, p_2) \\
(7) & \cdot\ x = [\lambda p_1 + (1-\lambda)q_1, \lambda p_2 + (1-\lambda)q_2] \\
 & \cdot\ q = (q_1, q_2)
\end{array}
$$

where λ is the ratio of the distance xq to pq. Using vector notation (this will be discussed more fully in § 8-2), the identical weighting of the corresponding coordinates of p and q may be written compactly $x = \lambda p + (1-\lambda)q$, which means

$$
\begin{aligned}
(8) \qquad x_1 &= \lambda p_1 + (1-\lambda)q_1 \qquad (0 \le \lambda \le 1) \\
x_2 &= \lambda p_2 + (1-\lambda)q_2
\end{aligned}
$$

To prove convexity for (5) we wish to show that x lies in R, which means its coordinates should satisfy $x_1 + x_2 \ge 2$ or to show that

$$(9) \qquad \lambda p_1 + (1-\lambda)q_1 + \lambda p_2 + (1-\lambda)q_2 \ge 2$$

To prove this we multiply the first inequality of (6) by $\lambda \ge 0$ and the second, by $1 - \lambda \ge 0$ to obtain

$$
\begin{aligned}
(10) \qquad \lambda p_1 + \lambda p_2 &\ge 2\lambda \\
(1-\lambda)q_1 + (1-\lambda)q_2 &\ge 2(1-\lambda)
\end{aligned}
$$

These two inequalities, when added together, result in (9), which establishes the convexity of R.

Convexity of Regions Defined by Linear Inequalities and Equations.

In n dimensions, the set of points whose coordinates satisfy a linear equation

$$(11) \qquad a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

is called a *hyperplane*, and the set of points whose coordinates satisfy a linear inequality such as

$$(12) \qquad a_1x_1 + a_2x_2 + \ldots + a_nx_n \le b$$

is called a *half-space* or to be precise, a *closed half-space* because we include the boundary. (In two dimensions it is called a *half-plane*.)

To prove the half-space defined by a linear inequality is convex, let p and q be any two points in the set, so that

$$(13a) \qquad a_1p_1 + a_2p_2 + \ldots + a_np_n \le b$$

$$(13b) \qquad a_1q_1 + a_2q_2 + \ldots + a_nq_n \le b$$

Let $0 \le \lambda \le 1$ be the value of the parameter associated with an arbitrary point x on the line segment joining p to q. Multiplying (13a) by $\lambda \geq 0$ and (13b) by $(1 - \lambda) \geq 0$ and adding, one obtains

$$(14) \quad a_1[\lambda p_1 + (1 - \lambda)q_1] + a_2[\lambda p_2 + (1 - \lambda)q_2] + \cdots + a_n[\lambda p_n + (1 - \lambda)q_n] \leq b$$

whence, substituting $x_i = \lambda p_i + (1 - \lambda)q_i$ by (3),

$$(15) \qquad a_1x_1 + a_2x_2 + \cdots + a_nx_n \leq b$$

Hence, an arbitrary point x on the line segment joining any two points lies in the half-space, establishing convexity.

To prove that a hyperplane is convex, let (11) be written as

$$(16) \qquad \begin{aligned} a_1x_1 + a_2x_2 + \cdots + a_nx_n &\leq b \\ a_1x_1 + a_2x_2 + \cdots + a_nx_n &\geq b \end{aligned}$$

Each of these inequalities defines a half-space and their intersection defines a hyperplane. Since a half-space is a convex set, then, by Theorem 1, a hyperplane is also a convex set. An n-dimensional space may contain many such convex sets. By Theorem 1, their common intersection is a convex set.

DEFINITION: A *convex polyhedron* is the set common to one or more half-spaces. In particular, a *convex polygon* is the intersection of one or more half-planes.

Convexity of the Set of Feasible and Optimal Feasible Solutions.

Consider now a general linear programming problem given by

$$(17) \qquad \begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \qquad (x_j \geq 0) \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

$$(18) \qquad c_1x_1 + c_2x_2 + \cdots + c_nx_n - z = 0$$

where z is to be minimized. We have just established

THEOREM 2: *The set of points corresponding to feasible* (*or optimal feasible*) *solutions of the general linear programming problem constitutes a convex set.*

Thus, if $p = (p_1, p_2, \ldots, p_n, z_p)$ is a feasible solution and $q = (q_1, q_2, \ldots, q_n, z_q)$ is another, the weighted linear combination of these two feasible solutions,

$$(19) \qquad [\lambda p_1 + (1 - \lambda)q_1, \ldots, \lambda p_n + (1 - \lambda)q_n; \lambda z_p + (1 - \lambda)z_q]$$

where λ is a constant, $0 \leq \lambda \leq 1$, is also a feasible solution. (This may be written compactly $x = \lambda p + (1 - \lambda)q$.) Moreover, assigning a fixed value for z, say $z = z_0$, the set of points satisfying (17), (18), and $z = z_0$ is also a convex set. In particular, setting $z_0 = \text{Min}\, z$, it is clear that the set of minimal feasible solutions is also a convex set.

A Local Minimum Solution Is Global.

In the calculus, the minimum (or maximum) of a function $f(x)$ with a continuous derivative is attained at a value x whose derivative is zero. This can result in a point like $x = x_1$ in Fig. 7-1-VII where $f(x)$ is minimum in the

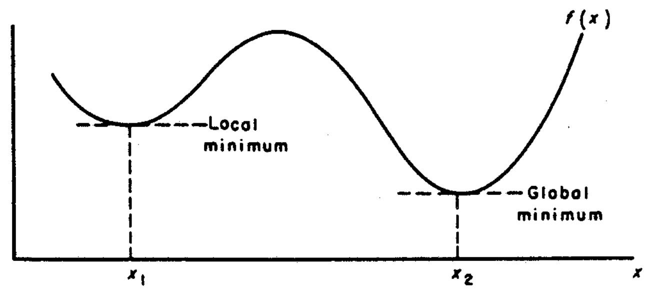


Figure 7-1-VII.

neighborhood of x_1; this is called a *local* minimum. However, it will also be noted there is another local minimum at $x = x_2$ where $f(x)$ attains its lowest value; this is called the *global* minimum. Any solution that is a local minimum solution is also a true (or global) minimum solution for the linear programming problem. To see this, let $p = (p_1, p_2, \ldots, p_n, z_p)$ be a local minimum solution and assume that it is not a true minimum solution, so that there is another solution $q = (q_1, q_2, \ldots, q_n, z_q)$ with $z_p > z_q$. Then any point $x = (x_1, x_2, \ldots, x_n, z)$ on a line segment joining these two points is a feasible solution and its $z = \lambda z_p + (1 - \lambda)z_q$. In this case the value of z decreases uniformly from z_p to z_q and thus all points on the line segment between p and q (including those in the neighborhood of p) have z values less than z_p and correspond to feasible solutions. Therefore, it is not possible to have a local minimum at p and at the same time another point q such that $z_p > z_q$. This means for all q, $z_p \leq z_q$, so that z_p is the true (global) minimum value.

DEFINITION: A function $f(x_1, x_2, \ldots, x_n)$ is a *convex function* if (1) it is defined over a set of points $p = (x_1, x_2, \ldots, x_n)$ which lie in a convex set C and if (2) the set of points in the one higher dimensional space $\bar{p} = (x_1, x_2, \ldots, x_n; z)$, where $z \geq f(x_1, x_2, \ldots, x_n)$, is a convex set $\bar{C}$.

For example, the function $f(x) = x^2$ is convex because the set of points (x, z) where $z \geq x^2$ is a convex set (see Fig. 7-1-VIII).

A Property of Convex Functions: If we let $x' = (x'_1, x'_2, \ldots, x'_n)$ and $x'' = (x''_1, x''_2, \ldots, x''_n)$ be any two points in the convex set C over which the convex function $f(x) = f(x_1, x_2, \ldots, x_n)$ is defined and x^* be any point on the segment joining x' to x'', so that $x^* = \lambda x' + (1 - \lambda)x''$ where $0 \leq \lambda \leq 1$, then

$$\lambda f(x') + (1 - \lambda) f(x'') \geq f(x^*) \tag{20}$$

For consider the two points $\bar{p}' = (x_1', x_2', \ldots, x_n'; z')$ and $\bar{p}'' = (x_1'', x_2'', \ldots, x_n''; z'')$ where $z' = f(x')$, $z'' = f(x'')$. The point $\bar{p}^* = (x_1^*, x_2^*, \ldots, x_n^*; z^*)$ where $z^* = \lambda z' + (1 - \lambda)z''$ lies in the convex set $\bar{C}$, and $z^* \geq f(x^*)$ because all points in the convex set $\bar{C}$ whose first n coordinates $x = x^*$ have a z coordinate greater or equal to $f(x^*)$ by definition. Geometrically (20) states that the z coordinate of $\bar{q} = [x_1^*, x_2^*, \ldots, x_n^*; f(x^*)]$ will never be higher than $\bar{p}^*$ if $f(x^*)$ is a convex function (see Fig. 7-1-VIII).

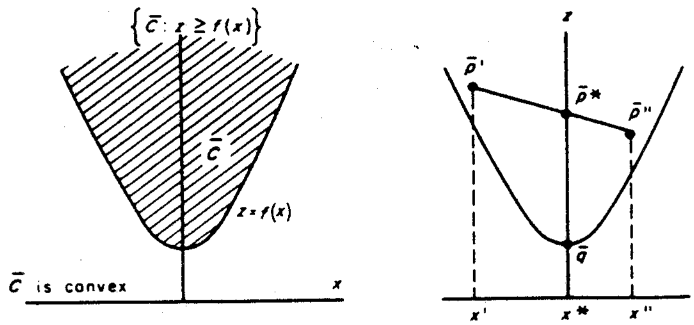


Figure 7-1-VIII. The curve $z = f(x)$ is called convex if $z \geq f(x)$ defines a convex set $\bar{C}$.

Exercise: Show that if the function $f(x)$ is not convex then (20) does not hold for at least two points x' and x'' in C.

Definition: Any point x in a convex set C which is not a midpoint of the line segment joining two other points in C is by definition an *extreme point* or *vertex* of the convex set. (Referring to Fig. 7-1-IX, the corners of

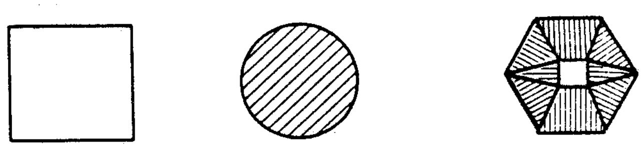


Figure 7-1-IX.

the square and every point on the circumference of a circle are extreme points. The points where three or more facets of a diamond come together are extreme points.)

Theorem 3: *A basic feasible solution corresponds to an extreme point in the convex set of feasible solutions.*

It is easy to show that a basic feasible solution corresponds to an extreme point. For example, suppose $x^o = (\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_m, 0, \ldots, 0)$ is a basic feasible solution for (17) with basic variables $x_1, x_2, \ldots, x_m$ and suppose it is the average of two other feasible solutions $p = (p_1, p_2, \ldots, p_m, \ldots, p_n)$ and $q = (q_1, q_2, \ldots, q_m, \ldots, q_n)$. Then

$$\tfrac{1}{2}(p_j + q_j) = 0$$

for all j corresponding to independent variables. But $p_j \geq 0$ and $q_j \geq 0$ because p and q are feasible solutions to (17). This is possible only if $p_j = q_j = 0$ for $j = m + 1, \ldots, n$. Thus p, q, and x^o have the same values (namely zero) for their independent variables. But the values of the basic variables are uniquely determined by the values of the independent variables and hence must be the same also. This shows $p = q = x^o$ and proves that x^o cannot be the average of two solutions p and q different from x^o.

DEFINITION: An *edge* of a convex polyhedron C is the straight line segment joining two extreme points such that no point on the segment is the midpoint of two other points in C not on the segment; in this case the two extreme points are said to be *neighbors* or *adjacent* to each other.

THEOREM 4: *The class of feasible solutions generated by increasing the value of a non-basic variable and adjusting the values of the basic variables in the change from one basic solution to the next corresponds to a movement along an edge of the convex set.*

PROOF: Suppose $p = (\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_m; 0, 0, \ldots, 0)$ is one basic feasible solution and $q = (0; \bar{b}_2^*, \ldots, \bar{b}_m^*, \bar{b}_{m+1}^*; 0, 0, \ldots, 0)$ is a basic feasible solution found by replacing x_1 in the basic set by, say, x_{m+1}. It is clear that any point $u = \lambda p + (1 - \lambda)q$ on the segment joining p to q has $u_{m+2} = u_{m+3} = \ldots = u_n = 0$. Hence, if u is to be the midpoint of two points p' and q' which are in the convex of feasible solutions, these components of p' and q' must also vanish. This permits one to express each of the first m components of p' and q' as a linear function of the value of the $(m + 1)^{st}$ component of p' and q', respectively. In fact, for any point x in the convex C whose components $x_{m+2} = x_{m+3} = \ldots = x_n = 0$ and x_{m+1} is arbitrary, we have

$$x_i = \bar{b}_i - \bar{a}_{i\,m+1}x_{m+1} \qquad (i = 1, 2, \ldots, m); \tag{21}$$

in particular, we have for $q = (0; \bar{b}_2^*, \bar{b}_3^*, \ldots, \bar{b}_m^*, \bar{b}_{m+1}^*; 0, \ldots, 0)$ that

$$\bar{b}_i^* = \bar{b}_i - \bar{a}_{i\,m+1}\bar{b}_{m+1}^* \qquad (i = 1, 2, \ldots, m) \tag{22}$$

Multiplying (22) by $\lambda = x_{m+1}/\bar{b}_{m+1}^*$ and subtracting from (21) yields

$$\begin{aligned} x_i &= \lambda\bar{b}_i^* + (1 - \lambda)\bar{b}_i && (i = 1, 2, \ldots, m) \\ x_{m+1} &= \lambda\bar{b}_{m+1}^* + (1 - \lambda)0 && \\ x_j &= \lambda 0 + (1 - \lambda)0 && (j = m + 2, \ldots, n) \end{aligned} \tag{23}$$

This proves that any two points, p' and q' in C, whose midpoint is u on the line segment joining p and q, are also on the line joining p and q. The assumption that p and q are extreme points implies $0 \leq \lambda \leq 1$, so that p' and q' are on the line segment joining p to q, which proves the line segment joining p and q forms an edge.

[Tucker, 1955-1] is recommended as collateral reading for this section.

7-2. THE SIMPLEX METHOD VIEWED AS THE STEEPEST DESCENT ALONG EDGES

Using a Set of Independent Variables as Coordinates of a Point in n − m Dimensions.

Consider a linear programming problem with $n = m + 2$ that has a basic feasible solution with respect to some m basic variables, say $x_3, x_4, \ldots, x_{m+2}$. The canonical form with respect to these variables is

$$
\begin{array}{llr}
\bar{a}_{11} x_1 + \bar{a}_{12} x_2 + x_3 & = \bar{b}_1 & \\
\bar{a}_{21} x_1 + \bar{a}_{22} x_2 \qquad + x_4 & = \bar{b}_2 & \\
\cdots\cdots\cdots\cdots\cdots\cdots & & \\
\bar{a}_{m1} x_1 + \bar{a}_{m2} x_2 \qquad\qquad + x_{m+2} & = \bar{b}_m & \\
\bar{c}_1 x_1 + \bar{c}_2 x_2 & = z - \bar{z}_0 &
\end{array} \tag{1} \qquad (\bar{b}_i \geq 0)
$$

where the problem is to find $x_j \geq 0$ and Min z satisfying (1). This is equivalent to finding values of x_1 and x_2 and the smallest constant $\bar{c}_0 = z - \bar{z}_0$ satisfying the system of linear inequalities

$$
\begin{array}{l}
x_1 \geq 0 \\
x_2 \geq 0 \\
\bar{a}_{11} x_1 + \bar{a}_{12} x_2 \leq \bar{b}_1 \\
\bar{a}_{21} x_1 + \bar{a}_{22} x_2 \leq \bar{b}_2 \\
\cdots\cdots\cdots\cdots \\
\bar{a}_{m1} x_1 + \bar{a}_{m2} x_2 \leq \bar{b}_m \\
\bar{c}_1 x_1 + \bar{c}_2 x_2 = \bar{c}_0
\end{array} \tag{2}
$$

We may graph these $m + 2$ relations in the two-dimensional space of the non-basic or independent variables x_1 and x_2 as illustrated in Fig. 7-2-I.

The convex region K formed by the half-spaces (in this case half-planes) $\bar{a}_{i1}x_1 + \bar{a}_{i2}x_2 \leq \bar{b}_i$ is shown by the solid lines in Fig. 7-2-I. The optimum solution is found by moving the dotted line $\bar{c}_1x_1 + \bar{c}_2x_2 = \bar{c}_0$ parallel to itself until the line just touches the convex and $\bar{c}_0$ is minimum. (If $\bar{c}_1$ and $\bar{c}_2$ are both less than zero this would be in the direction away from the origin.) Associated with every point P in K is a unique feasible solution to (1). In fact such a point P must satisfy all the inequalities (2) and the nonnegative difference between the values on the left hand side of (2) and the right hand side are the unique values of the basic variables in (1) when the non-basic variables x_1 and x_2 have the specified values (x_1^o, x_2^o). The value $x_{i+2} = x_{i+2}^o$ of the i^{th} basic variable is proportional to the distance of the point $P = (x_1^o, x_2^o)$ from the boundary of the i^{th} constraint because, from analytic

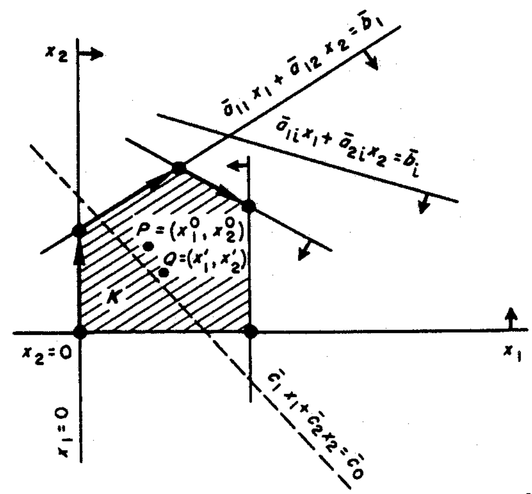


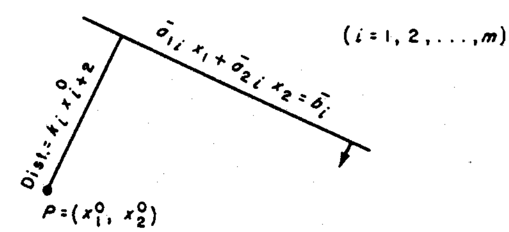
Figure 7-2-I. Geometrically the simplex algorithm moves along edges of the convex.

geometry, the *distance* of P from $\bar{a}_{i1}x_1 + \bar{a}_{i2}x_2 = \bar{b}_i$ is given by (3) for $i = 1, 2, \ldots, m$,

(3) $$\text{distance} = \frac{\bar{b}_i - \bar{a}_{i1}x_1^o - \bar{a}_{i2}x_2^o}{[\bar{a}_{i1}^2 + \bar{a}_{i2}^2]^{\frac{1}{2}}} = k_i x_{i+2}^o$$

If the point (x_1^o, x_2^o) satisfies the inequality, then the geometric picture is

(4)



where $k_i = (\bar{a}_{i1}^2 + \bar{a}_{i2}^2)^{-\frac{1}{2}}$.

If the variables are replaced by $y_i = k_i x_{i+2}$ for $i = 1, 2, \ldots, m$, and the coordinates of a point P are the values of the independent variables, then *the value of the* i^{th} *basic variable is just the distance from the point* P *to the corresponding* i^{th} *constraint.*

Every basic solution to (1) has at least two $x_j = 0$, hence the corresponding P is at the same time a point in K and is at zero distance to two distinct boundary lines of K. It is intuitively evident (and we show this rigorously below) that such a P is a vertex of K. In particular, the basic feasible solution with respect to the canonical form (1) is associated with the point $(x_1^o = 0, x_2^o = 0)$ in Fig. 7-2-I, hence the origin is always in the convex K.

We now show in a little more rigorous manner that *associated with every extreme point in the convex set of feasible solutions to* (1) *is an extreme point of* K *and conversely.* To this end, let $P = (x_1^o, x_2^o)$ and $Q = (x_1', x_2')$ be any two points in K, and let the corresponding feasible solutions satisfying (1) be $p = (x_1^o, x_2^o, \ldots, x_n^o)$ and $q = (x_1', x_2', \ldots, x_n')$ which as we have seen in Theorem 2 of § 7-1 lie in a convex set C. It is easy to see that any point $\lambda P + (1 - \lambda)Q$ on the line joining P to Q corresponds to a point $\lambda p + (1 - \lambda)q$ that satisfies (1), and conversely. Hence line segments in the convex C of solutions satisfying (1) correspond to line segments in K, and in particular the midpoint of a segment in C corresponds to the midpoint in K and conversely. It follows that non-extreme points must correspond to each other and it must logically follow that extreme points (basic feasible solutions) to (1) correspond to extreme points of K and conversely.

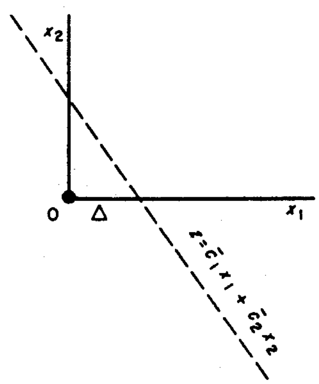
Moreover, *the movement along the edge* corresponding to the class of feasible solutions generated by increasing a non-basic variable and adjusting the values of the basic variables in the shift from one basic solution to the next, must correspond to a movement around the boundary of K from one vertex to the next. To see this, let p and q be successive distinct extreme points corresponding to basic feasible solutions obtained by the simplex method under non-degeneracy, so that the line segment joining p to q is an edge in C. If now the corresponding vertices P and Q in K were not neighbors, there would be a point X on the segment joining P to Q that would be the midpoint of two points P' and Q' in K, but not on the segment. We shall show, however, that P' and Q' must lie on the line joining P to Q. We have shown that x, corresponding to X must be the midpoint of p' and q' corresponding to P' and Q'. However, x must also be on the line joining p to q since X was on the line joining P to Q. It follows since the segment pq is an *edge* (§ 7-1, Theorem 4), p' and q' must both be on this edge and hence their corresponding points P' and Q' must lie on the line joining P to Q. This shows that edges in the convex of feasible solutions to (1), correspond to edges in Fig. 7-2-I.

Thus the simplex method proceeds from one vertex to the next in the space of a fixed set of non-basic variables. Starting with the vertex at the origin and moving successively from one neighboring vertex to another, each step decreases the value of $\bar{c}_0$ until a minimum value for $\bar{c}_0$ is obtained as shown by the arrows in Fig. 7-2-I.

The General Case.

While our remarks have been restricted to the case of $n = m + 2$ for simplicity, they hold equally well for the general case of $n = m + k$. In this case, the values of $k = n - m$ of any set of non-basic variables become the coordinates of a point in k dimensions. In this geometry the convex K of feasible solutions is defined by a set of m inequalities formed by dropping the basic variables in the canonical form and adding the k inequalities $x_j \geq 0$ where x_j are the non-basic variables. Each basic feasible solution corresponds to a vertex of K. In the general (non-degenerate) situation, there are $n - m$ edges from each vertex leading to $n - m$ neighboring vertices; these correspond to the $n - m$ basic solutions obtained by introducing one of the $n - m$ non-basic variables in place of one of the basic variables. *The simplex criterion of choosing $\bar{c}_s = \text{Min } \bar{c}_j < 0$ followed by an increase in x_s corresponds to a movement along that edge of the convex which induces the greatest decrease in z per unit change in the variable introduced.*

(5)



For example, for $n = m + 2$, see (5), if $\bar{c}_1 < \bar{c}_2$, then any movement for a distance Δ along the x_1-axis produces a greater decrease in z than an equal movement along the x_2-axis.

It can be shown in general that the simplex method is a steepest descent "gradient" technique in which the "gradient direction" is defined in the space of non-basic variables, say $x_{m+1}, x_{m+2}, \ldots, x_n$. Translating the origin to some trial solution point, the *usual* steepest gradient direction is defined by finding the limiting direction as $\Delta \to 0$ from this origin to a point on the *spherical* surface

(6) $$x_{m+1}^2 + x_{m+2}^2 + \ldots + x_n^2 = \Delta^2 \qquad (x_j \geq 0)$$

where the function z is minimized. In contradistinction, the simplex algorithm's steepest gradient direction is found using a *planar* surface instead of a spherical surface

$$x_{m+1} + x_{m+2} + \dots + x_n = \Delta \qquad (x_j \geq 0) \tag{7}$$

In other words, in defining the gradient, the usual (Euclidean) distance (6) from the origin (located at some trial solution point) is replaced by one based on the sum of the absolute values of the coordinates (7).

EXERCISE: Consider the problem of minimizing $\bar{c}_{m+1}x_{m+1} + \bar{c}_{m+2}x_{m+2} + \dots + \bar{c}_n x_n$ subject to (7) for fixed Δ where $x_j \geq 0$. Show that the solution is to choose $x_s = \Delta$ and all other $x_j = 0$ where $\bar{c}_s = \text{Min}\ \bar{c}_j$. What is the steepest gradient direction as $\Delta \to 0$?

EXERCISE: Consider the problem of minimizing $\bar{c}_{m+1}x_{m+1} + \bar{c}_{m+2}x_{m+2} + \dots + \bar{c}_n x_n$ subject to (6) for fixed Δ where x_j is unrestricted in sign. Show that the solution is to choose $x_j = -\bar{c}_j\theta$ where $\theta = \Delta^2/\Sigma\bar{c}_j^2$. What is the steepest gradient direction as $\Delta \to 0$?

7-3. THE SIMPLEX INTERPRETATION OF THE SIMPLEX METHOD

While the simplex method appears a natural one to try in the n-dimensional space of the variables, it might be expected, *a priori*, to be inefficient as there could be considerable wandering on the outside edges of the convex of solutions before an optimal extreme point is reached. This certainly appears to be true when $n - m = k$ is small, such as in Fig. 7-2-I where $k = 2$. However, empirical experience with thousands of practical problems indicates that the number of iterations is usually close to the number of basic variables in the final set which were not present in the initial set. For an m-equation problem with m different variables in the final basic set, the number of iterations may run anywhere from m as a minimum, to $2m$ and rarely to $3m$. The number is usually less than $3m/2$ when there are less than 50 equations and 200 variables (to judge from informal empirical observations). Some believe that for a randomly chosen problem with fixed m, the number of iterations grows in proportion to n.

It has been conjectured that, by proper choice of variables to enter the basic set, it is possible to pass from any basic feasible solution to any other in m or less pivot steps, where each basic solution generated along the way must be feasible. For the cases $m \leq 4$ the conjecture is known to be true. [W. M. Hirsch, 1957, verbal communication.]

Moreover, when the simplex method is viewed in the m-dimensional space associated with the columns of coefficients of the variables, as will be done in this section, the method appears to be quite efficient. It was in this geometry that the method was first seriously proposed after it had been earlier set aside as unpromising.

In Chapter 3, both the Blending Model II and the Product Mix

Model were graphically solved using as the coordinates of a point the coefficients of a variable in one of the equations and the cost form. For this purpose it was assumed that one of the equations of the model could be written in the form

(1) $$x_1 + x_2 + \ldots + x_n = 1 \qquad (x_j \geq 0 : \pi_0)$$

leaving, for the case $m = 2$, one other equation and cost form

(2) $$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b \qquad (: \pi_1)$$

(3) $$c_1x_1 + c_2x_2 + \ldots + c_nx_n = z \text{ (Min)}$$

The variables x_j were interpreted as nonnegative weights to be assigned to a system of points $A_j = (a_j, c_j)$ in two-dimensional space (u, v) so that their weighted average (center of gravity) is a point $R = (b, \text{Min } z)$; that is to say the $x_j \geq 0$ are chosen so that the center of gravity lies on the "requirement line" $u = b$ (constant) and such that the v coordinate is minimum (see Fig. 7-3-I).

Convex Hull.

In Fig. 7-3-I, the shaded area C represents the set of all possible centers of gravity G formed by assigning different weights x_j to the points A_j. It is

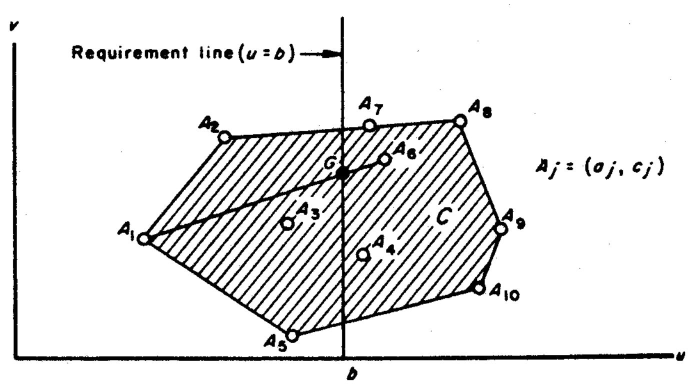


Figure 7-3-I. Geometrically a linear program is a center of gravity problem.

easy to prove these form a convex region C, called the *convex hull* of the set of points A_j. To see this, let G' be any point in C obtained by using nonnegative weights $w_1', w_2', \ldots, w_n'$ and G'' any other point obtained by using nonnegative weights $w_1'', w_2'', \ldots, w_n''$. Let $G^* = \lambda G' + (1 - \lambda)G''$, where $0 \leq \lambda \leq 1$, be any point on the line segment joining G' to G''. G^* must lie in C also because it can be obtained by using weights $w^* = \lambda w_j' + (1 - \lambda)w_j''$ for $j = 1, 2, \ldots, n$; moreover, if $w_j' \geq 0$, $w_j'' \geq 0$, $\Sigma w_j' = 1$, $\Sigma w_j'' = 1$ and $0 \leq \lambda \leq 1$, then $w_i^* \geq 0$, $\Sigma w_j^* = 1$. This establishes the convexity of C.

It is also easy to see that any column (activity) corresponding to a point A_j which is not an extreme point of the convex hull can be dropped from the linear programming problem. Thus the points A_3, A_4, A_6 in the *interior* of C in Fig. 7-3-I and A_7 on an *edge* can be dropped; that is to say, one can set $x_3 = x_4 = x_6 = x_7 = 0$ and still obtain a feasible solution with just as low a minimum value.

A basic feasible solution corresponds to a pair of points, say A_1 and A_6 in Fig. 7-3-I, such that the line joining A_1 to A_6 intersects the constant line $u = b$ in a point G on the line segment between A_1 and A_6. For this to be true we would want

$$\lambda a_1 + (1 - \lambda)a_6 = b_1 \qquad (0 \leq \lambda \leq 1)$$

But this corresponds to the basic feasible solution to (1) and (2) found by setting $x_1 = \lambda$, $x_6 = (1 - \lambda)$ and $x_j = 0$ for all other j.

To improve the solution, the simplex method first computes the relative cost factors $\bar{c}_j$ by eliminating the basic variables from the cost equation. We shall now show that this is the same as first computing the line joining A_1 to A_6, which we will refer to as the *solution line*, and then substituting the coordinates of a point A_j into the equation of the line to see how much (if any) in the v-direction it is *above* or *below* the line; see Fig. 7-3-II.

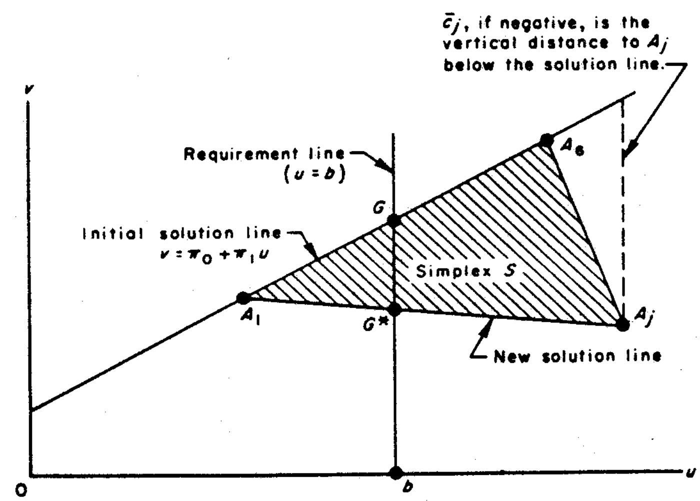


Figure 7-3-II. The simplex associated with a cycle of the simplex algorithm ($m = 2$).

To eliminate basic variables x_1 and x_6 from (3), suppose equation (1) is multiplied by π_0 and equation (2) by π_1 and subtracted from (3). Then π_0 and π_1 must be chosen so that

$$(4) \qquad c_1 - (\pi_0 + \pi_1 a_1) = 0$$

$$c_6 - (\pi_0 + \pi_1 a_6) = 0 \tag{5}$$

and the relative cost factors $\bar{c}_j$ are given by

$$\bar{c}_j = c_j - (\pi_0 + \pi_1 a_j) \tag{6}$$

Let us compare this with what would be required to compute the line

$$v = \pi_0 + \pi_1 u \tag{7}$$

where constants π_0 and π_1 are chosen so that the line passes through the points $A_1 = (a_1, c_1)$ and $A_6 = (a_6, c_6)$. Substituting $u = a_1$ and $v = c_1$ into equation (7) gives the condition that A_1 lies on this line, while substituting $u = a_6$, $v = c_6$ yields the condition for A_6 to be on this line. But these are precisely conditions (4) and (5). To determine *how much* a point with coordinates $u = a_j$, $v = c_j$ is above or below the solution line in the v-direction, we first determine the ordinate of the point where the line $u = a_j$ cuts $v = \pi_0 + \pi_1 u$, namely at $v = \pi_0 + \pi_1 a_j$, and then subtract this value from the ordinate c_j of A_j, denoted by $\bar{c}_j$ in (6). Thus A_j is *above*, *on*, or *below* the line according as $\bar{c}_j > 0$, $\bar{c}_j = 0$, or $\bar{c}_j < 0$.

The condition that a basic feasible solution is minimal is that $\bar{c}_j \geq 0$ for all non-basic variables c_j. *Geometrically* it states that a *basic feasible solution is optimal* if all points A_j lie on or above its solution line. For example, in Fig. 7-3-I, the requirement line $u = b$ cuts the line segment joining A_5 to A_{10}, and all other points A_j lie above the extended line joining these two points; hence the minimal solution is obtained by using x_5 and x_{10} as basic variables.

On the other hand, if there is a point A_j, as in Fig. 7-3-II, *below* the solution line, then join A_j to A_1 and to A_6 and consider the convex figure S formed by $A_1A_6A_j$. This is the *convex hull of three points in* $m = 2$ *dimensions and is called a two-dimensional simplex*. If A_j is below the solution line, every point of S is also. If G is not at a vertex, there is a segment G^*G on the requirement line belonging to S below the solution line with G^*, the lowest point. Thus there exists a *new solution line* passing through G^*. It is either A_1A_j or A_6A_j depending on whether A_j is on the right or left of $u = b$.

In Fig. 7-3-III, we illustrate the steps of the simplex method geometrically on (8) the Product Mix Problem, § 3-5.

$$\begin{array}{l} y_1 + \quad y_2 + \quad y_3 + \quad y_4 + \quad y_5 + \quad y_6 = 1 \qquad (y_j \geq 0) \\ \left\{ \begin{array}{l} .2y_1 + .1y_2 + .3y_3 + .8y_4 + 0y_5 + 1y_6 = .4 \\ -2.4y_1 - 2.0y_2 - 1.8y_3 - .8y_4 + 0y_5 + 0y_6 = z \text{ (Min)} \end{array} \right. \end{array} \tag{8}$$

Let the coordinates of a point A_j in Fig. 7-3-III be the coefficients of y_j in the second and third equations:

$$A_1 = (.2, -2.4), \quad A_2 = (.1, -2.0), \quad A_3 = (.3, -1.8),$$
$$A_4 = (.8, -.8), \quad A_5 = (0, 0), \quad A_6 = (1, 0)$$

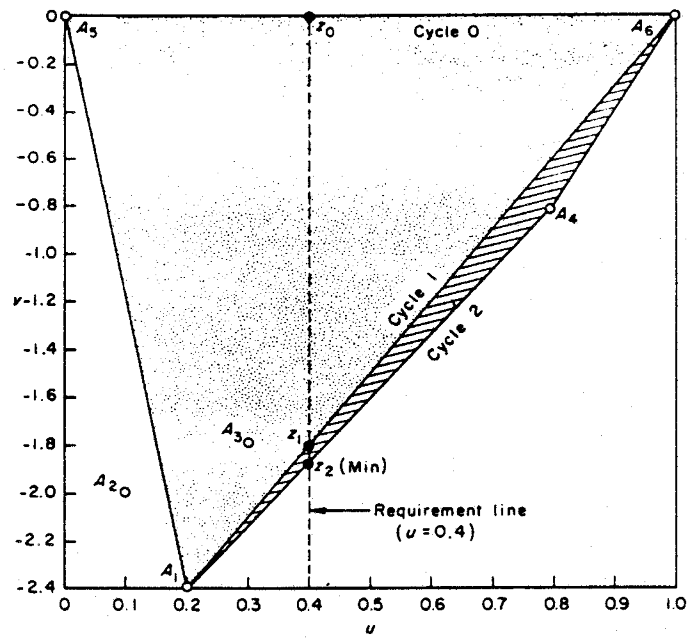


Figure 7-3-III. The simplex algorithm geometrically illustrated on the product mix problem.

The simplex iterations may be summarized as follows:

Iteration	Basic variables	Solution line	Simplex
0	y_5, y_6	A_5A_6	$> A_5A_6A_1$
1	y_1, y_6	A_1A_6	$> A_4A_6A_1$
2	y_1, y_4	A_1A_4	

Simplex Defined.

In higher dimensions, say m, the convex hull of $m + 1$ points in general position (see definition below) is called an *m-dimensional simplex*; thus

0-dim. simplex is a point
1- ,, ,, ,, a line segment
2- ,, ,, ,, a triangle and its interior
3- ,, ,, ,, a tetrahedron and its interior

DEFINITION: Let $A_j = (a_{1j}, a_{2j}, \ldots, a_{mj})$ be the coordinates of a point A_j in m-dimensional space. A set of $m + 1$ points $[A_1, A_2, \ldots, A_{m+1}]$

is said to be a *general position* if the determinant of their coordinates and a row of ones, as in (9), is non-vanishing,

$$(9)\qquad \begin{vmatrix} 1 & 1 & \ldots 1 \\ a_{11} & a_{12} & \ldots a_{1,m+1} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \ldots a_{m,m+1} \end{vmatrix} \neq 0$$

For $m = 3$ dimensions, consider the problem of finding $x_j \geq 0$ and Min z satisfying

Multipliers

$$(10)\qquad x_1 + x_2 + \ldots + x_n = 1 \quad (x_j \geq 0) \qquad : \pi_0$$

and

$$(11)\qquad \begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 & : \pi_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 & : \pi_2 \\ c_1x_1 + c_2x_2 + \ldots + c_nx_n &= z \end{aligned}$$

Define as coordinates (u_1, u_2, v) of a point the coefficients of x_j in (11); thus $A_j = (a_{1j}, a_{2j}, c_j)$. The *requirement line* is $u_1 = b_1$, $u_2 = b_2$. A basic feasible solution corresponds to three points, say A_1, A_2, A_3 such that the requirement line intersects the "solution plane" formed by A_1, A_2, A_3 at a point of the two-dimensional simplex formed by A_1, A_2, A_3 as in Fig. 7-3-IV. If

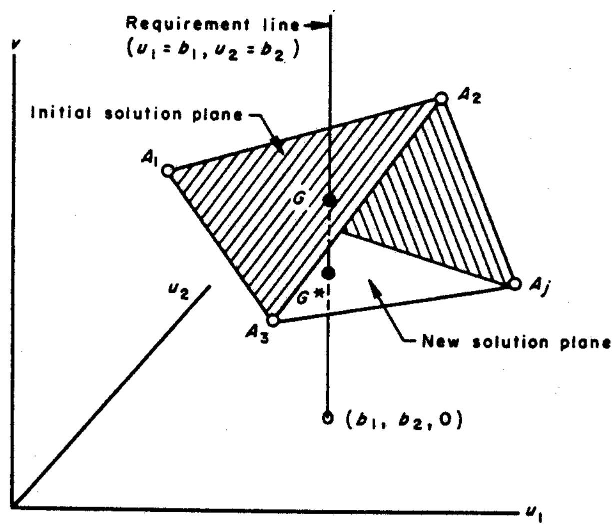


Figure 7-3-IV. The simplex associated with a cycle of the simplex algorithm ($m = 3$).

A_j is a point below the solution plane, $v = \pi_0 + \pi_1 u_1 + \pi_2 v_2$, associated with A_1, A_2, A_3, then $\bar{c}_j = c_j - \pi_0 - \pi_1 a_{1j} - \pi_2 a_{2j} < 0$. In this case, a three-dimensional *simplex* $A_j A_1 A_2 A_3$ can be formed and a point G^* found where the requirement line pierces the simplex at a lower point. G^* is on one of the three faces $A_1 A_2 A_j$, $A_2 A_3 A_j$, $A_1 A_3 A_j$ depending on the position of A_j. In Fig. 7-3-IV, G^* was assumed to be in the face $A_1 A_3 A_j$, and it is these three points that are used to determine the new solution plane.

The simplex criterion used to select a new basic variable x_s does not select an arbitrary x_j corresponding to an A_j below the solution plane, but an $A_s = A_j$ which is a *maximum* distance, $\bar{c}_s = \text{Min } \bar{c}_j$ below the plane. Inspection of figures such as Figs. 7-3-I and 7-3-II give credence to the belief that such a point would have a good chance of being in the optimal solution. Empirical evidence on thousands of problems confirms this and is the reason the simplex method is efficient in practice.

EXERCISE: Study Fig. 7-2-I and § 7-2-(5); construct an example to show for $n = m + 2$ that the simplex criterion $\bar{c}_s = \text{Min } \bar{c}_j$ could cause a maximum number of cycles to be performed.

7-4. PROBLEMS

Convex Regions. (Refer to § 7-1.)

1. Review the relationship between convex sets and linear programming.
2. Determine which of the following are convex sets.

(a)
$$\begin{aligned} x_1 + x_2 + x_3 &\geq 6 \\ x_1 - x_2 + 3x_3 &\geq 4 \\ x_j &\geq 0 \end{aligned}$$

(b) $x_1^2 + x_2^2 \geq 5$

(c) $x_1^2 + 2x_2^2 \leq 3$

(d) $x_2^2 - 2x_1 \leq 2$

(e)
$$\begin{aligned} x_1 - 2x_2^2 &\geq 3 \\ x_1 + 2x_2 &\leq 4 \\ 2x_1 + 3x_2 &\geq 6 \end{aligned}$$

3. (a) Solve graphically and by the Fourier-Motzkin Elimination Method of § 4-4: maximize $x_1 - x_2$, subject to

$$\begin{aligned} x_1 + x_2 &\leq 5 \\ x_1 - 3x_2 &\geq 0 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \\ 2x_1 + 3x_2 &\geq 6. \end{aligned}$$

(b) State in standard form.

(c) Indicate the convex set of feasible solutions. Why is the optimal solution an extreme point?

(d) Omitting the conditions $x_1 \geq 0$ and $x_2 \geq 0$, reduce to standard form by two methods.

4. Transform the two systems of inequalities (A) and (B) below into systems of equations in nonnegative variables by a change of variables; graph (B) in terms of the original variables. Graph the dual of (A).

$$(A) \begin{cases} 2x_1 + 3x_2 + 4x_3 \geq 5 \\ 4x_1 - 7x_2 + 3x_3 \leq 4 \end{cases} \qquad (B) \begin{cases} x_1 + x_2 \leq 1 \\ 4x_1 + 8x_2 \leq 32 \\ x_1 + x_2 \leq 4 \\ x_1 - 2x_2 \geq 2 \end{cases}$$

5. The process of increasing the variable x_s in the simplex method, while holding the other independent variable fixed at zero, generates a class of solutions corresponding to an edge in a convex polyhedron of feasible solutions if the vertex corresponds to a nondegenerate basic feasible solution. What can happen under degeneracy?
6. If a basic solution is nondegenerate, there are precisely $n - m$ neighbors of its corresponding extreme point and these are generated by increasing one of the $n - m$ independent variables, while holding the remainder fixed at zero. What can happen under degeneracy?
7. Show that if x_k is a variable unrestricted in sign, it is possible to obtain an optimal solution for the system by eliminating x_k from all but one equation, setting this equation aside and optimizing the remaining modified system, and then determining x_k by a back substitution.
8. Suppose that one equation of a linear program in standard form has one positive coefficient, say that of x_k, while all remaining coefficients of the equation are nonpositive and the constant is positive. This implies that $x_k > 0$ for any solution, whatever the values of the remaining $x_j \geq 0$. Pivot on any non-zero term in x_k to eliminate x_k from the remaining equations and set aside the one equation involving x_k. Prove that the resulting linear program in one less equation and one less variable can be solved to find the optimal solution of the original problem.
9. If it is known in advance that a solution cannot be optimal unless it involves a variable x_k at positive value, show that this variable can be eliminated and the reduced system with one less equation and variable solved in its place.
10. Devise a method for finding the second best basic feasible solution. Generalize to the third best, the fourth, etc. Discuss any complications.
11. Show, if r variables have unique and nonnegative values when the remaining variables are set equal to zero, the feasible solution is an extreme point solution.
12. Given an extreme point solution $(v_1, v_2, \ldots, v_n)$, show that if the

variables x_j are set equal to zero corresponding to $v_j = 0$, then the remaining variables are uniquely determined and $x_j = v_j > 0$.

13. (W. M. Hirsch, unsolved.) Does there exist a sequence of m or less pivot operations, each generating a new basic feasible solution (b.f.s.), which starts with some given b.f.s. and ends with some other given b.f.s., where m is the number of equations? Expressed *geometrically*:

    In a convex region in $n - m$ dimensional space defined by n half-planes, is m an upper bound for the minimum-length chain of adjacent vertices joining two given vertices?
14. Prove that a square homogeneous system of m equations always has a nontrivial solution (a solution in which at least one variable is not zero) if there are redundant equations (i.e., if the rank of the system is less than m).
15. (Gale.) Prove that a square homogeneous linear inequality system always has a nontrivial solution.
16. Show that the set of possible values of any variable x_k of a linear program forms a convex set, in this case, a straight line segment $a \leq x_k \leq b$.
17. Show that the set of possible values of two variables, say (x_1, x_2) or (x_1, z) satisfying a linear program, forms a convex set in two dimensions.
18. As a corollary to problem 17, show, if x_k is treated as a parameter and can take on a range of possible values, that the value of Min z becomes a convex function of x_k.
19. Show, in general, that Min z is a convex function of θ, if the constant terms of a linear program are linear functions of the parameter θ. However, show that the value of some other variable, such as x_4 for Min z in the example below, need not be either a convex or a concave (the negative of a convex) function of θ.

$$
\begin{array}{llllllll}
x_1 & & -x_3 & +x_4 & & & = \theta & \quad (x_j \geq 0)\\
x_1 & + x_2 & & & & & = \theta & \\
x_1 & & -x_3 & & +x_5 & & = 1 & \\
 & + x_2 & & & & +x_6 & = 1 & \\
4x_1 & + 2x_2 & & +x_4 & & & = z \text{ (Min)} &
\end{array}
$$

20. Show that, if $P = (a_1, a_2, \ldots, a_m)$ is a point in m dimensions, the set of points C with coordinates $a_1\lambda, a_2\lambda, \ldots, a_m\lambda$, where λ can take on any value in the range $0 \leq \lambda < \infty$, is convex. This set is called a *ray*. Graph the ray for $P = (1, 1, 1)$.
21. A set of points is called a *cone* if, whenever P is in the set, so is every point in the ray of P. Construct an example to show that a cone, in general, need not be convex.
22. Show that, if $P = (a_1, a_2, \ldots, a_m)$ and $Q = (a'_1, a'_2, \ldots, a'_m)$ are two points in m dimensions, the set of points C with coordinates $X = (\lambda a_1 + \mu a'_1, \lambda a_2 + \mu a'_2, \ldots, \lambda a_m + \mu a'_m)$ for arbitrary λ and μ in

the ranges $0 \leq \lambda < \infty$, $0 \leq \mu < \infty$ forms a convex cone. In vector notation (see § 8-2), X is given by $X = \lambda P + \mu Q$. This set is called the cone generated by two rays λP and μQ. Graph the cone generated by $P = (1, 1, 1)$, and $Q = (1, 1, 0)$.

23. In general, the set generated by forming nonnegative linear combinations of points $P_1, P_2, \ldots, P_k$ is called a cone. Thus, C is all points $X = \lambda_1 P_1 + \lambda_2 P_2 + \ldots + \lambda_k P_k$ for arbitrary λ_i in the range $0 \leq \lambda_i < \infty$. Prove C is a convex cone.
24. Show that a convex cone is formed by the set C of all points $P^* = (b_1, b_2, \ldots, b_m)$ given by choosing $x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0$ in the expressions

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n = b_i \qquad (i = 1, 2, \ldots, m)$$

25. Suppose $P_1, P_2, \ldots, P_k, \ldots$ is an infinite collection of points in m dimensional space. Let C be the set of points generated by forming nonnegative linear combinations of finite subset of these points. Let C' be the set of points generated by forming nonnegative linear combinations of subset of m or less of these points. Show that C and C' are identical convex cones.

Interpretations of the Simplex Method. (Refer to §§ 7-2, 7-3.)

26. Carry out the steps of the simplex method both algebraically and geometrically on (a) the Product Mix Problem and (b) the Blending Problem II and show the correspondence. (Refer to § 3-4 and § 3-5 for the problems.)
27. Take the warehouse problem (§ 3-6) and solve algebraically and geometrically using the simplex method in three dimensions.
28. Using the Fourier-Motzkin elimination procedure, solve

$$\begin{aligned} 2y_1 + y_2 &\leq 2 \\ -3y_1 + y_2 &\leq -3 \\ y_1 - 2y_2 &\leq 6 \\ 3y_1 + 9y_2 &\leq 1 \\ -y_1 \qquad &\leq -2 \\ 3y_1 + 4y_2 &= v \text{ (Max)} \end{aligned}$$

29. Solve the above, using the following variant of the simplex method: for those with positive right-hand sides introduce *slack* variables $y_j \geq 0$; for those with nonpositive right-hand sides introduce *artificial excess* variables $y_j \geq 0$. Apply the usual simplex method to minimizing the sum of artificial variables, in this case $y_4 + y_7 = w$. However, note that y_1 and y_2 are not restricted in sign. See problem below.
30. Invent a variant of the simplex method which permits specified variables to be unrestricted in sign. Apply this to Problem 28.

31. Solve

$$\begin{aligned} 2y_1 + \ y_2 &\leq 2 \\ y_1 - 2y_2 &\leq 6 \\ 3y_1 + 9y_2 &\leq 1 \\ 3y_1 + 4y_2 &= v \text{ (Max)} \end{aligned} \qquad (y_1 \geq 0, y_2 \geq 0)$$

using the simplex method. Interpret geometrically the simplex steps in the 2-dimensional space of y_1 and y_2.

32. Given a system

$$\begin{aligned} x_1 + \ x_2 + \ldots + \ x_n &= 1 \\ a_1x_1 + a_2x_2 + \ldots + a_nx_n &= b \\ c_1x_1 + c_2x_2 + \ldots + c_nx_n &= z \text{ (Min)} \end{aligned} \qquad (x_j \geq 0)$$

show that the solution line $v = \pi_0^* + \pi_1^*u$ associated with the minimal basic solution must satisfy

$$\begin{aligned} &\text{(a)} \quad c_j - (\pi_0^* + \pi_1^*a_j) \geq 0 \\ &\text{(b)} \quad \pi_0^* + \pi_1^*b = \text{Min } z \end{aligned}$$

33. Prove in the above that the convex hull of points $A_j = (a_j, c_j)$ lies above an arbitrary line $v = \pi_0 + \pi_1u$, if

$$c_j - (\pi_0 + \pi_1a_j) \geq 0$$

Use this to show that such a line must cut the requirement line $u = b_1$ in a point, whose ordinate $v \leq \text{Min } z$.

34. Use the results of the above two problems to prove that the values of π_0 and π_1 and Max v, satisfying the (dual) inequality system

$$\begin{aligned} \pi_0 + \pi_1a_j &\leq c_j \qquad \text{for } j = 1, 2, \ldots, n \\ \pi_0 + \pi_1b &= v \text{ (Max)} \end{aligned}$$

are given by the solution line $v = \pi_0^* + \pi_1^*u$ associated with the minimal basic solution and

$$\text{Max } v = \text{Min } z$$

Review this particular geometrical interpretation of the duality theorem given in § 3-4 and displayed in Fig. 3-4-I.

35. Note that the dual of a standard linear program is a system of inequalities in unrestricted variables. Suppose one is given a system in the latter form; review how its dual may be used as a *third way* to get a standard linear program from a system of linear inequalities. Find the standard linear program of which this is the dual:

$$\begin{aligned} \pi_0 + .2\pi_1 &\leq -2.4 \\ \pi_0 + .1\pi_1 &\geq -2.0 \\ \pi_0 + .3\pi_1 &\leq -1.8 \end{aligned}$$

$$\begin{aligned} \pi_0 + .8\pi_1 &\le -.8 \\ \pi_0 \qquad &\le 0 \\ \pi_0 + \pi_1 &\le 0 \\ \pi_0 \qquad &\le v \text{ (Max)} \end{aligned}$$

Solve the dual, by using the simplex method and also by using the elimination method, and prove that Max $v =$ Min z of the dual original system.

36. If $v = \pi_0 + \pi_1 u_1 + \pi_2 u_2$ represents the solution plane associated with A_1, A_2, A_3 in Fig. 7-3-III, interpret the conditions

$$v_j - (\pi_0 + \pi_1 a_{1j} + \pi_2 a_{2j}) = 0 \qquad (j = 1, 2, 3)$$

and the quantities

$$v_j - (\pi_0 + \pi_j a_{1j} + \pi a_{2j}) = \bar{c}_j$$

both algebraically in the simplex method and geometrically.

37. A third geometry of the simplex method can be obtained by regarding a column j as representing a *line* $\pi_0 + a_j \pi_1 = c_j$ in (π_0, π_1) space. Thus, this procedure can be interpreted to be in the same space as the space of independent variables π_0 and π_1 of the *dual linear programming problem* $\pi_0 + a_j \pi_1 \le c_j$, $\pi_0 + b\pi_1 = v$ (Max) for $j = 1, 2, \ldots, n$. Show that the simplex procedure for solving the dual is different from the interpretation of the simplex procedure for solving the original problem in this geometry. (The procedure of Kelley for solving nonlinear programs is based on this geometry.) See [Wolfe, 1960-1].

38. In the text the relation between the classical gradient procedure and the simplex method is outlined. Show that each iteration of the simplex method expresses the function to be minimized in terms of a different set of independent variables. Show that the direction of maximum decrease of the function under the restrictions $\Sigma x_j = \Delta$, $x_j \ge 0$ is just the one given by the simplex criterion. What would it be if $\Sigma x_j^2 = \Delta^2$ were used instead?

39. (a) Use the "Center of Gravity Method" to find $x_j \ge 0$ and Min z satisfying

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 4x_4 &= z \text{ (Min)} \\ x_1 + x_2 + x_3 + x_4 &= 4 \\ x_1 - 2x_2 + 3x_3 - 4x_4 &= -2 \end{aligned}$$

(b) Dualize and graph the dual problem.

(c) Solve the dual using the Fourier-Motzkin Elimination Method (§ 4-4).

(d) Solve the primal using the simplex method. Trace the steps of the procedure as graphed in (a) and (b).

40. [Minkowski, 1896-1.] *Theorem:* A feasible solution of a bounded linear program can be expressed as a linear nonnegative combination of

basic feasible solutions. Geometrically stated, a point of a bounded convex polyhedron C, defined as the intersection of finitely many half-spaces, can be expressed as a linear nonnegative combination of extreme points of C.

Show that the theorem is false if C is unbounded.

Advanced Problems.

41. *Theorem:* Let M be a given set of points in a Euclidean $(m-1)$ dimensional space and let Q be in the convex hull of M. It is possible to find m points $P_1, P_2, P_3, \ldots, P_m$ (not necessarily different) of M, and m real numbers $x_1 \ldots x_m$ so that $x_i \geq 0$, $\sum_1^m x_i = 1$, and $\sum_1^m x_i P_i = Q$. (E. Steinitz, *Reine Angew. Math.*, Vol. 143, 1913, pp. 128–275.)
42. *Theorem:* Let M be a given infinite set of points in Euclidean m-dimensional space and let Q be in the convex cone spanned by M. It is possible to find m points $P_1, P_2, \ldots, P_m$ (not necessarily different) of M, and m real numbers $x_1 \geq 0, \ldots, x_m \geq 0$, so that $\sum_1^m x_i P_i = Q$.

    Hint: Establish this theorem for any point Q representable as a nonnegative finite linear combination of points $P_i \in$ M. Show that all such points Q define the convex cone spanned by M.

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CHAPTER 8

PIVOTING, VECTOR SPACES, MATRICES, AND INVERSES

8-1. PIVOT THEORY[1]

Our purpose is to extend the discussion of § 4-2 regarding properties preserved by pivot operations and characteristics of pivot operations. The first of five important properties concerns redundancy and inconsistency.

THEOREM 1: *If there is a linear combination of equations of a system with non-zero weights which results in a null equation (redundancy), or in an inconsistent equation, then the same is true for a system obtained from it by a sequence of elementary (or pivot) operations.*

PROOF: Let E_0 represent alternatively either a vacuous or an inconsistent equation. Let $E_1, E_2, \ldots, E_k$ denote a subset of the equations of the system that, by the hypothesis, satisfy

$$\lambda_1 E_1 + \lambda_2 E_2 + \cdots + \lambda_k E_k = E_0 \qquad \text{where } \lambda_i \neq 0 \tag{1}$$

If the first of a sequence of elementary operations does not involve these E_i, the same relation will hold in the resulting system. The same is clearly true if an elementary operation replaces E_1 by kE_1, say, where $k \neq 0$. If E_1 is replaced by $E_1 + kE_2 = E_1'$, then $E_1 = E_1' - kE_2$ and the relation

$$\lambda_1 E_1' + (\lambda_2 - \lambda_1 k)E_2 + \cdots + \lambda_k E_k = E_0 \tag{2}$$

holds for the resulting system. Since $\lambda_1 \neq 0$ in this case too, a non-zero linear combination of the equations of the resulting system yields E_0. Finally, if E_1, say, is replaced by $E_1 + kE_t = E_1'$ where $t \neq 1, 2, \ldots, k$, then the relation

$$\lambda_1 E_1' + \lambda_2 E_2 + \cdots + \lambda_k E_k - \lambda_1 k E_t = E_0 \tag{3}$$

holds, and again the result follows since $\lambda_1 \neq 0$. By induction the theorem holds for any number of elementary operations. Since pivot operations are a particular case of the latter, the proof is complete.

The second important property of the pivot operation is its *irreversibility* except in certain situations.

[1] I am indebted to A. W. Tucker for his suggestions for developing this section based on the idea of the irreversibility of the pivot operations except when applied to a canonical form.

THEOREM 2: *If a system is in canonical form before a pivot operation, then it is in canonical form after the pivot operation.*

PROOF: Suppose a system is in canonical form with basic variables $x_{j_1}, x_{j_2}, \ldots, x_{j_m}$. Let the pivot term be chosen in equation r using variable x_s. Then, as we have seen in § 5-1, the resulting system is in canonical form with x_s replacing x_{j_r} as a basic variable. In the new system, if a new pivot term is selected in equation r using variable x_r, then a second pivot operation will restore the original system; hence

COROLLARY 1: *If a system is in canonical form, the inverse of a pivot operation is a pivot operation.*

COROLLARY 2: *If a subsystem S is in canonical form before a pivot operation and if the pivot term is selected in an equation of S, then the corresponding subsystem after a pivot operation is in canonical form; the inverse of the pivot operation is a pivot operation (assuming zero coefficients for basic variables in the non-canonical equations).*

COROLLARY 3: *If a subsystem S is in canonical form before a pivot operation and if the pivot term is selected in an equation E not in S, then the subsystem corresponding to $\{S, E\}$ after a pivot operation is in canonical form; the inverse of the pivot operation is not a pivot operation unless $\{S, E\}$ was in canonical form initially.*

The third important property of the pivot operation is that there is a one-to-one correspondence between equations and that easily defined subsets of the original and the derived systems are equivalent.

DEFINITION: The *pivotal subsystem* is that set of equations P of the *original system* corresponding to those selected for pivot terms in a sequence of pivot operations.

It is clear that the number of equations in the pivotal subsystem increases or remains the same during a sequence of pivot operations, depending on whether or not the successive pivot terms are selected from among equations corresponding to the pivotal system or from among the remainder. Let S be any subset of the original equations that includes the pivotal set P and let S' and P' be the corresponding subsets after a sequence of pivot operations.

THEOREM 3: *The system S' is equivalent to S; in particular, P' is equivalent to P and, moreover, P' is in canonical form.*

PROOF: That P' is canonical, follows from Corollaries 2 and 3. To prove S and S' are equivalent systems, note that if the equations not in S are deleted, the same sequence of pivot operations can be performed on those corresponding to S and hence the latter are all equivalent to S.

THEOREM 4: *The pivotal subsystem P is independent and solvable.*

PROOF: P' is in canonical form by Theorem 3 and is therefore solvable and independent. Since P is equivalent to P', it is solvable also. It cannot contain any redundancies because by Theorem 1 the same would have to hold for P'.

THEOREM 5: (*A redundancy and inconsistency tracing theorem.*) *If an equation* E'_t *of a reduced system is vacuous* (*or inconsistent*), *then in the original system,* E_t *is either redundant with respect to the pivotal system* P (*or a linear combination of the equations of* P *and* E_t *can form an inconsistent equation*).

PROOF: Note $\{P, E_t\}$ can be generated from $\{P', E'_t\}$ by a reverse sequence of elementary operations, hence applying Theorem 1 there exist weights $\lambda_i \geq 0$ not all zero such that

$$\lambda_1 E_1 + \lambda_2 E_2 + \ldots + \lambda_k E_k + \lambda_t E_t = E_0 \qquad (\lambda_i \geq 0) \tag{4}$$

where $(E_1, \ldots, E_k)$ are the pivotal equations (not necessarily the first k) and E_0 is alternatively either a vacuous or an inconsistent equation. In either case, $\lambda_t \neq 0$ because of Theorems 3 and 4; hence, if E_0 is vacuous, E_t is dependent on the others.

Testing Systems for Equivalence.

The fourth important property of the pivoting operation is that it provides a way to show whether or not two systems have *the same solution set* by trying to reduce them simultaneously by pivoting step by step, using the *same* pivotal variables. The same process will test the *equivalence* of two systems.

THEOREM 6: *Two solvable systems have the same solution sets if and only if they are equivalent.*

THEOREM 7: *Two systems are equivalent if and only if it is possible to pivot with respect to the same ordered sequence of variables and* (a) *if consistent, the canonical parts of the two systems are identical and the remainder vacuous;* (b) *if inconsistent, the canonical parts are identical except possibly for the constant terms, and the remainder each have one or more inconsistent equations.*

PROOF OF THEOREMS: Let us suppose first that it is possible to reduce two systems using the same set of pivot variables. We assume the equations of the canonical parts are reordered so that both subsystems are canonical with the same set of basic variables. If the two systems are to be *equivalent,* it is necessary that their canonical part be identical, because there is only *one* way to form the left-hand side of an equation of the canonical part of one system as a linear combination of the equations of the reduced system of the other. Their constant terms may not agree if there are inconsistent equations in the non-canonical part (but may be made to agree by adding in a suitable multiple of the latter). If the two systems are solvable with the *same solution set*, the canonical parts are identical; (see proof of Theorem 1 in § 4-2). In general, the non-canonical parts must either both contain an inconsistent equation or both be vacuous because the only way to generate an inconsistent equation of one system from that of the other is as a linear combination of the inconsistent equations of the other.

Now let us suppose that it is *not* possible to reduce two systems using the same set of pivot variables, but that it is possible to pivot on the same variables for the first t steps, say variables $x_1, x_2, \ldots, x_t$ in the first t equations, and that on step $(t + 1)$ it is possible to pivot on the x_{t+1} term in equation $t + 1$ in system I, and it is not possible to use x_{t+1} for pivotal variable in system II because the coefficients of x_{t+1} are zero in *all* the remaining $m - t$ equations of II. Note that it is not possible to generate the $t + 1^{st}$ equation of system I from those of system II because the weights on the first t equations must be zero and this makes the coefficient of x_{t+1} automatically zero whatever be the weights on the remaining equations. Hence the two systems cannot be equivalent.

Nor can the two systems, in this case, be solvable with the same solution set. To see this, let $x_1^o, \ldots, x_r^o, x_{r+1}^o, \ldots, x_n^o$ be *any* solution to system I. Either it does not satisfy system II, or if it does, then a solution for system II exists which does not satisfy system I, namely $x_1^*, \ldots, x_r^*; x_{r+1}^*, x_{r+2}^o, \ldots, x_n^o$ obtained by changing x_{r+1}^o to $x_{r+1}^* \neq x_{r+1}^o$ and adjusting the values of $x_1, \ldots, x_r$ in canonical part of the first r equations of system II. Note that this solution satisfies the remaining equations of system II (there are no $x_1, \ldots, x_r, x_{r+1}$ terms) but cannot satisfy system I because it does not satisfy equation $r + 1$ of system I. Hence, in either case, the two systems do not have the same solution set.

The fifth important property of pivoting is that it provides a way to prove a number of interesting theorems concerning the number of independent and dependent equations of a system.

THEOREM 8: *Two equivalent, independent, consistent systems have the same number of equations.*

PROOF: Invoking Theorem 4 it is possible simultaneously to reduce the two systems, and the canonical parts of the reduced systems are identical. No vacuous equations can result because pivoting is actually a sequence of elementary operations, so that, by Theorem 1, the appearance of such equations would imply a redundant equation in the original systems. Therefore, the identical canonical equivalents have the same number of equations as their respective original systems.

The following three theorems are consequences of the above.

THEOREM 9: *Two equivalent canonical systems have the same number of equations.*

THEOREM 10: *If a system has a canonical equivalent with r equations, any partition of the system into an independent set of equations and a set of equations dependent upon them will have exactly r equations in the independent set.*

THEOREM 11: *If a system has a canonical equivalent with r equations, then any r independent equations of the system can generate the remainder by linear combinations.*

DEFINITION: The largest number of independent equations in a solvable system is called its *rank*.

Exercise: Prove Theorems 9, 10, and 11. Show that r in Theorem 10 is the rank of the system.

8-2. VECTOR SPACES

Vector Operations.

Many operations that are performed on a system of equations can be viewed as performing a number of operations *in parallel*. For example, we may rewrite the system,

$$\begin{aligned} 2x_1 + 3x_2 - 4x_3 &= 5 \\ -4x_1 - 2x_2 + 3x_3 &= 7 \end{aligned} \tag{1}$$

in the form

$$\begin{bmatrix} 2 \\ -4 \end{bmatrix} x_1 + \begin{bmatrix} 3 \\ -2 \end{bmatrix} x_2 + \begin{bmatrix} -4 \\ +3 \end{bmatrix} x_3 = \begin{bmatrix} 5 \\ 7 \end{bmatrix} \tag{2}$$

and interpret this to mean that when the corresponding elements (components) in the column are to be multiplied by the unknowns and added across, their sums give the corresponding elements in the right-hand column. The columns are called *column vectors*. Operations, called "addition" and "scalar multiplication" of vectors, are performed upon them in a manner analogous to ordinary numbers.

The coefficients [2, 3, −4] that appear in the first equation (or [−4, −2, 3] in the second equation) may likewise be considered as an entity called a *row vector*. Vectors whose elements are drawn from a row are usually written with brackets [ ] or parentheses ( ). Often vectors whose elements are from a column are written in text as row vectors to conserve space; when this is the case for us, angle parentheses $\langle\ \rangle$ will be used instead of [ ] or ( ). Thus $\langle 2, -4 \rangle$ stands for the column vector $\begin{bmatrix} 2 \\ -4 \end{bmatrix}$.

Definition: An m-vector is an ordered set of m numbers called components (elements).

We shall begin by defining two fundamental operations on vectors which are a natural extension of addition and multiplication of numbers to sets of numbers in parallel.

Definition: The *scalar multiple* of an m-vector by a number (scalar) x is an m-vector formed by multiplying each component by x. Thus for a column vector,

$$\begin{bmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ a_m \end{bmatrix} x = \begin{bmatrix} a_1x \\ a_2x \\ \cdot \\ \cdot \\ \cdot \\ a_mx \end{bmatrix} \tag{3}$$

DEFINITION: The *sum* of two m vectors is the vector formed by adding the corresponding components. Thus

$$\begin{bmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ a_m \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_m \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \cdot \\ \cdot \\ \cdot \\ a_m + b_m \end{bmatrix} \tag{4}$$

DEFINITION: Two m-vectors are equal if their corresponding components are equal. Thus

$$\begin{bmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ a_m \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_m \end{bmatrix} \text{ means } a_i = b_i \text{ for } i = 1, 2, \ldots, m \tag{5}$$

With this interpretation of operations on vectors it is clear that (2) is the same as (1) because, by the scalar multiplication of vectors, (2) is the same as

$$\begin{bmatrix} 2x_1 \\ -4x_1 \end{bmatrix} + \begin{bmatrix} 3x_2 \\ -2x_2 \end{bmatrix} + \begin{bmatrix} -4x_3 \\ +3x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix} \tag{6}$$

and by addition of vectors (say, by adding the third vector to the sum of the first two)

$$\begin{bmatrix} 2x_1 + 3x_2 - 4x_3 \\ -4x_1 - 2x_2 + 3x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix} \tag{7}$$

and by equality of vectors (7) means (1).

We are now in a position to make the important observation that the so-called *elementary operations on equations are in essence the scalar multiplication and addition of the row vectors formed by detaching the coefficients and constant terms of the equations*. The variables play a passive role throughout. For example, if the first equation of (1) is multiplied by 2 and added to the second, we obtain $0x_1 + 4x_2 - 5x_3 = 17$. This corresponds to the operations $2[2, 3, -4, 5] + [-4, -2, 3, 7] = [0, 4, -5, 17]$.

Linearly Dependent Vectors.

A vector each of whose components is zero is called a *zero vector* (or *null vector*). Thus by a vector $V = 0$ is meant

$$V = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \tag{8}$$

A vector $V \neq 0$ means that at least one component of V differs from zero.

A vector $\langle y_1, y_2, \ldots, y_m\rangle$ is said to be *linearly dependent* on n other vectors $P_j = \langle a_{1j}, a_{2j}, \ldots, a_{mj}\rangle$ if one can find numbers (scalars) $x_1, x_2, \ldots, x_n$, such that

$$(9) \qquad \begin{bmatrix} a_{11} \\ a_{21} \\ \cdot \\ \cdot \\ \cdot \\ a_{m1} \end{bmatrix} x_1 + \begin{bmatrix} a_{12} \\ a_{22} \\ \cdot \\ \cdot \\ \cdot \\ a_{m2} \end{bmatrix} x_2 + \ldots + \begin{bmatrix} a_{1n} \\ a_{2n} \\ \cdot \\ \cdot \\ \cdot \\ a_{mn} \end{bmatrix} x_n = \begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_m \end{bmatrix}$$

EXERCISE: Choose particular values of a_{ij} and y_i, such that there are no x_j that satisfy (9); choose values, such that there are unique x_j that satisfy (9); choose values, such that there are many sets of x_j that satisfy (9).

DEFINITION: A set of r vectors P_i is *linearly independent* if

$$(10) \qquad P_1x_1 + P_2x_2 + \ldots + P_rx_r = 0$$

implies $x_1 = x_2 = \ldots = x_r = 0$. If a set of vectors is *not* linearly independent, then (10) holds with at least one $x_i \neq 0$ and the set is said to be *linearly dependent*. It is easy to see that this P_i is linearly dependent on the others.

EXERCISE: Show that the set consisting of a single vector is an independent set unless it is the zero vector. Given any set of vectors show that the null vector is linearly dependent upon them.

An m-vector whose i^{th} element is unity and all other elements are zero is called a *unit vector*. The m *different* unit vectors are denoted by

$$(11) \qquad U_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, U_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \ldots, U_m = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix}$$

EXERCISE: Show that the vectors U_i are linearly independent. Show that any other vector can be expressed as a linear combination of the unit vectors U_i.

Vector Equations.

If, as above, we use symbols to denote vectors, we can write a single vector equation to represent m linear equations. For example, let

$$(12) \qquad Q = \begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_m \end{bmatrix}; \; P_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \cdot \\ \cdot \\ \cdot \\ a_{mj} \end{bmatrix} \qquad (j = 1, 2, \ldots, n)$$

Then (9) becomes the problem of determining weights x_j (if possible) which express a linear dependence between the vectors P_j and Q,

$$P_1x_1 + P_2x_2 + \cdot \cdot \cdot + P_nx_n = Q \tag{13}$$

Vector Space.

Instead of seeking numbers x_j that satisfy (13), we may reverse the process and generate column vectors $Q = \langle y_1, y_2, \cdot \cdot \cdot, y_m \rangle$ by varying the values $x_1, x_2, \cdot \cdot \cdot, x_n$. The set of vectors $\langle y_1, y_2, \cdot \cdot \cdot, y_m \rangle$ generated by all possible choices of $(x_1, x_2, \cdot \cdot \cdot, x_n)$ is called a *vector space*.

For example, if we plot in two dimensions the points with coordinates (y_1, y_2) obtained by choosing different values of x_1 and x_2 in (14), it is clear that it will describe the entire (y_1, y_2) plane.

$$\begin{bmatrix}1\\2\end{bmatrix} x_1 + \begin{bmatrix}2\\1\end{bmatrix} x_2 = \begin{bmatrix}y_1\\y_2\end{bmatrix} \tag{14}$$

On the other hand, the points (y_1, y_2) of (15) lie on the line $2y_1 = y_2$.

$$\begin{bmatrix}1\\2\end{bmatrix} x_1 = \begin{bmatrix}y_1\\y_2\end{bmatrix} \tag{15}$$

The vector space for (y_1, y_2, y_3) in (16) is the plane $y_3 = y_1 + y_2$ in 3 dimensions,

$$\begin{bmatrix}1\\2\\3\end{bmatrix} x_1 + \begin{bmatrix}2\\1\\3\end{bmatrix} x_2 = \begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix} \tag{16}$$

It is easy to see that the points (y_1, y_2, y_3) in (17) also lie in the plane $y_3 = y_1 + y_2$, because the column vectors associated with x_3 and x_4 are linearly dependent on those corresponding to x_1 and x_2.

$$\begin{bmatrix}1\\2\\3\end{bmatrix} x_1 + \begin{bmatrix}2\\1\\3\end{bmatrix} x_2 + \begin{bmatrix}3\\3\\6\end{bmatrix} x_3 + \begin{bmatrix}-1\\+1\\0\end{bmatrix} x_4 = \begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix} \tag{17}$$

In fact, substituting in (17) the expressions

$$\begin{bmatrix}3\\3\\6\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix} + \begin{bmatrix}2\\1\\3\end{bmatrix}, \quad \begin{bmatrix}-1\\+1\\0\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix} - \begin{bmatrix}2\\1\\3\end{bmatrix} \tag{18}$$

one obtains

$$\begin{bmatrix}1\\2\\3\end{bmatrix} (x_1 + x_3 + x_4) + \begin{bmatrix}2\\1\\3\end{bmatrix} (x_2 + x_3 - x_4) = \begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix} \tag{19}$$

and it is clear that the class of vectors $\langle y_1, y_2, y_3 \rangle$ generated by (19) is no smaller than that generated by (16).

DEFINITION: A *basis* of a vector space is any set of independent vectors in the space such that all other vectors in the space can be generated as linear combinations of the vectors in the set.

It is easy to see that there can be many sets of independent vectors that can generate the same vector space. Thus the vector spaces associated with (14) and with (20) below are the same.

$$\begin{bmatrix}1\\0\end{bmatrix} x_1 + \begin{bmatrix}0\\1\end{bmatrix} x_2 = \begin{bmatrix}y_1\\y_2\end{bmatrix} \tag{20}$$

Rank or Dimensionality of a Vector Space.

The rank of a vector space is the largest number of independent vectors in the space. We shall show that, if a vector space can be generated from r independent vectors, any other set of r independent vectors in the space can also serve as a basis. Moreover, it is not possible to generate the space with fewer than r vectors nor is it possible to find in the space more than r independent vectors. The number r is called the rank or dimensionality of the vector space.

THEOREM 1: *Let Q be any vector in the vector space generated by a set of independent vectors $(P_1, P_2, \ldots, P_r)$; then the values $x_1, x_2, \ldots, x_r$ such that*

$$P_1x_1 + P_2x_2 + \ldots + P_rx_r = Q \tag{21}$$

are unique.

PROOF: If not unique then there exists another set of values x_1' such that

$$P_1x_1' + P_2x_2' + \ldots + P_rx_r' = Q \tag{22}$$

Subtraction yields

$$P_1(x_1 - x_1') + P_2(x_2 - x_2') + \ldots + P_r(x_r - x_r') = 0 \tag{23}$$

and we conclude that if not all $(x_i - x_i') = 0$, the vectors $P_1, P_2, \ldots, P_r$ are not independent, contrary to assumption.

DEFINITION: The expression (21) is called the *representation* of the vector Q in terms of the basis $(P_1, P_2, \ldots, P_r)$, and $(x_1, x_2, \ldots, x_r)$ are called the *coordinates* of Q relative to this basis.

EXERCISE: Show that the set of unit vectors (11) constitute a basis in the space E_m of all vectors with m coordinates, and the coordinates of a vector relative to this basis are the same as the components of the vector.

THEOREM 2: *Given a basis and a vector $R \neq 0$ in a vector space, it is possible to replace one of the columns of the basis by R to form a new basis.*

PROOF: Let the representation of R in terms of the basis be

$$P_1v_1 + P_2v_2 + \ldots + P_rv_r = R \tag{24}$$

At least one $v_i \neq 0$ in (24), since $R \neq 0$. Suppose $v_1 \neq 0$; then we will show that a new basis can be formed by replacing P_1 by R. First of all,

$P_2, P_3, \ldots, P_r; R$ are linearly independent; for, assuming they are linearly dependent implies that R has a non-zero coefficient and thus can be expressed in terms of the others in a representation different from the *unique* representation (24), a contradiction (see Theorem 1).

Now we only need to show that an arbitrary Q can be expressed in terms of the independent vectors $P_2, P_3, \ldots, P_r; R$ to prove these vectors form a basis. In fact, multiplying (24) by an arbitrary constant θ and subtracting from (21) yields

$$(25) \quad P_1(x_1 - \theta v_1) + P_2(x_2 - \theta v_2) + \ldots + P_r(x_r - \theta v_r) + R\theta = Q$$

whence setting

$$(26) \qquad \theta = x_1/v_1 \qquad (v_1 \neq 0)$$

shows that Q is linearly dependent upon the others, since $x_1 - \theta v_1 = 0$. Hence, these independent vectors can generate any other vector Q in the space.

THEOREM 3: *Given a basis and k independent non-zero vectors $R_1, R_2, \ldots, R_k$, in the vector space generated by the basis, it is possible to replace k vectors in the basis by $R_1, R_2, \ldots, R_k$.*

PROOF: The proof is inductive. The case $k = 1$ was shown by the previous theorem. Suppose that a new basis can be formed by substituting $k - 1$ vectors $R_1, R_2, \ldots, R_{k-1}$ for $k - 1$ vectors in the basis, say, by replacing $P_1, P_2, \ldots, P_{k-1}$, so that the new basis is $R_1, R_2, \ldots, R_{k-1}; P_k, \ldots, P_r$. Let the representation of R_k in terms of this basis be

$$(27) \quad R_1v_1 + R_2v_2 + \ldots + R_{k-1}v_{k-1} + P_kv_k + \ldots + P_rv_r = R_k$$

At least one $v_i \neq 0$, for $i \geq k$, otherwise R_k would be linearly dependent on $R_1, R_2, \ldots, R_{k-1}$, contrary to assumption. Let $v_t \neq 0$ for some $t \geq k$. Then, following the argument of the previous theorem, R_k can replace P_t in this basis to form a new basis consisting of vectors $R_1, R_2, \ldots, R_{k-1}, R_k, P_{k+1}, \ldots, P_r$ (omitting P_t). The following are left as exercises:

THEOREM 4: *If there exists a basis consisting of r vectors, then any r independent vectors in the vector space form a basis.*

THEOREM 5: *If there exists a basis consisting of r vectors, then it is not possible to have more than r independent vectors in the vector space.*

THEOREM 6: *If there exists a basis consisting of r vectors, then it is not possible to find in the vector space a basis with fewer than r vectors.*

EXERCISE: Show that the symbolic operations on equations E_i in § 8-1 may also be viewed as vector relations. Let $\bar{E}_i = (a_{i1}, a_{i2}, \ldots, a_{in}; b_i)$ be the row vector, defined by the coefficients and constant of E_i; then § 8-1-(1) may be interpreted to mean

$$\lambda_1\bar{E}_1 + \lambda_2\bar{E}_2 + \lambda_3\bar{E}_3 + \ldots + \lambda_k\bar{E}_k = \bar{E}_0$$

Interpret the other symbolic relations in § 8-1.

EXERCISE: Show that the rank of a consistent system of equations $E_1, E_2, \ldots$ is the same as the rank of the system of row vectors $\bar{E}_1, \bar{E}_2, \ldots$ associated with these equations. The definition for equations is given at the end of § 8-1.

8-3. MATRICES

Matrix Operations.

A rectangular array of numbers is called a matrix. Thus the detached coefficients of § 8-2-(1)

$$\begin{bmatrix} 2 & 3 & -4 \\ -4 & -2 & 3 \end{bmatrix} \tag{1}$$

constitute a 2×3 matrix, i.e., a matrix of two rows and three columns. More generally an $m \times n$ matrix is

$$\begin{bmatrix} a_{11} & a_{12} \cdots a_{1n} \\ a_{21} & a_{22} \cdots a_{2n} \\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ a_{m1} & a_{m2} \cdots a_{mn} \end{bmatrix} = A = [a_{ij}] \tag{2}$$

which may be denoted by a single letter, such as A, or by $[a_{ij}]$, where a_{ij} is the symbol for the value in row i and column j. The *transpose* of the matrix A is denoted by A' or A^T and is obtained by interchanging rows and columns. If $A = [a_{ij}]$ and $A^T = [b_{ij}]$, then $b_{ji} = a_{ij}$.

DEFINITION: Two $m \times n$ matrices are *equal* if all corresponding elements are equal. Thus,

$$[a_{ij}] = [b_{ij}] \text{ means } a_{ij} = b_{ij} \quad (i = 1, 2, \ldots, m; j = 1, 2, \ldots, n) \tag{3}$$

DEFINITION: The *sum* of two $m \times n$ matrices is the $m \times n$ matrix formed by adding the corresponding elements. Thus,

$$[c_{ij}] = [a_{ij}] + [b_{ij}] \tag{4}$$

means that for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$,

$$c_{ij} = a_{ij} + b_{ij} \tag{5}$$

For example,

$$\begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 4 \\ 6 & 6 & 8 \end{bmatrix} \tag{6}$$

It is through the concept of the *multiplication* of two vectors that a significant generalization of operations on numbers is achieved. The basic idea is to consider

$$2x_1 + 3x_2 - 4x_3 \tag{7}$$

as the product of two vectors (2, 3, −4) and (x_1, x_2, x_3). The convention is to make one of them a row vector and the other a column vector with the row vector preceding the column vector.

DEFINITION: The *(scalar) product* of a row vector by a column vector, each of n components, is a number (scalar) equal to the sum of the products of corresponding components; i.e.,

$$(8) \qquad [a_1, a_2, \ldots, a_n]\begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} = a_1x_1 + a_2x_2 + \ldots + a_nx_n$$

If $n = 1$, (8) becomes ordinary multiplication of two numbers.

DEFINITION: The *product* of an $m \times n$ matrix A by an n-vector X is

$$(9) \qquad AX = \begin{bmatrix} a_{11} & a_{12} \ldots & a_{1n} \\ a_{21} & a_{22} \ldots & a_{2n} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} \ldots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ \cdot \quad \cdot \quad \ldots \quad \cdot \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \end{bmatrix}$$

This definition supersedes § 8-2-(3) and (8) above since the former is obtained if $n = 1$, and the latter is obtained if $m = 1$. According to (9), the product AX is a vector, the i^{th} component of which is the product of the i^{th} row of A (considered as a row vector) by the column vector X. Indeed, if A_i denotes the i^{th} row of the matrix (2), i.e.,

$$(10) \qquad A_i = (a_{i1}, a_{i2}, \ldots, a_{in}) \qquad (i = 1, 2, \ldots, m)$$

the matrix A may be viewed as a column of row vectors

$$(11) \qquad A = \begin{bmatrix} a_{11} & a_{12} \ldots & a_{1n} \\ a_{21} & a_{22} \ldots & a_{2n} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} \ldots & a_{mn} \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \\ \cdot \\ \cdot \\ \cdot \\ A_m \end{bmatrix}$$

Now letting X symbolize an n-vector,

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} \tag{12}$$

it is seen that, analogous to the multiplication of a column vector by a scalar § 8-2-(3), the multiplication of a matrix by a vector (9) is defined as

$$AX = \begin{bmatrix} A_1 \\ A_2 \\ \cdot \\ \cdot \\ \cdot \\ A_m \end{bmatrix} X = \begin{bmatrix} A_1X \\ A_2X \\ \cdot \\ \cdot \\ \cdot \\ A_mX \end{bmatrix} \tag{13}$$

A matrix may also be viewed as a row of column vectors. Thus, if P_j denotes the j^{th} column of (2), i.e.,

$$P_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \cdot \\ \cdot \\ \cdot \\ a_{mj} \end{bmatrix} \qquad (j = 1, 2, \ldots, n) \tag{14}$$

then

$$A = \begin{bmatrix} a_{11} & a_{12} \ldots a_{1n} \\ a_{21} & a_{22} \ldots a_{2n} \\ \cdot & \cdot\cdot\cdot\cdot\cdot \\ a_{m1} & a_{m2} \ldots a_{mn} \end{bmatrix} = [P_1, P_2, \ldots, P_n] \tag{15}$$

Therefore, analogous to the multiplication of a row vector by a column vector (8), the product of a matrix by a vector (9) is given by

$$AX = [P_1, P_2, \ldots, P_n] \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} = P_1x_1 + P_2x_2 + \ldots + P_nx_n \tag{16}$$

since

$$(17)\quad P_1x_1 + P_2x_2 + \ldots + P_nx_n$$

$$= \begin{bmatrix} a_{11} \\ a_{21} \\ \cdot \\ \cdot \\ \cdot \\ a_{m1} \end{bmatrix} x_1 + \begin{bmatrix} a_{12} \\ a_{22} \\ \cdot \\ \cdot \\ \cdot \\ a_{m2} \end{bmatrix} x_2 + \ldots + \begin{bmatrix} a_{1n} \\ a_{2n} \\ \cdot \\ \cdot \\ \cdot \\ a_{mn} \end{bmatrix} x_n$$

$$= \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ \cdot \qquad \cdot \qquad \ldots \qquad \cdot \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \end{bmatrix}$$

We can also define the product of a row vector $\pi = (\pi_1, \pi_2, \ldots, \pi_m)$ of m components by an $m \times n$ matrix by analogy to vectors (see § 8-2-(3)). Thus, we would expect

$$(18)\qquad \pi A = \pi[P_1, P_2, \ldots, P_n] = [\pi P_1, \pi P_2, \ldots, \pi P_n]$$

DEFINITION: The *product* of a row vector π of m components by an $m \times n$ matrix is a row vector whose j^{th} component is the product of π by the j^{th} column of A (considered as a column vector). Thus

$$(19)\quad [\pi_1, \pi_2, \ldots, \pi_m] \begin{bmatrix} a_{11} & a_{12} \ldots a_{1n} \\ a_{21} & a_{22} \ldots a_{2n} \\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ a_{m1} & a_{m2} \ldots a_{mn} \end{bmatrix} = [\pi P_1, \pi P_2, \ldots, \pi P_n]$$

where

$$\begin{aligned} \pi P_1 &= \pi_1 a_{11} + \pi_2 a_{21} + \ldots + \pi_m a_{m1} \\ \pi P_2 &= \pi_1 a_{12} + \pi_2 a_{22} + \ldots + \pi_m a_{m2} \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ \pi P_n &= \pi_1 a_{1n} + \pi_2 a_{2n} + \ldots + \pi_m a_{mn} \end{aligned}$$

This definition supersedes § 8-2-(3) which is obtained by setting $m = 1$; when $n = 1$, it agrees with (8) which is a special case of (9). However, it is possible to generalize the definition of product once more and make both (9) and (19) special cases of the following:

Definition of multiplication of matrices: The *product* of an $(m \times k)$ matrix A by a $(k \times n)$ matrix $\bar{A}$ is an $(m \times n)$ matrix $A\bar{A}$ whose element in row i, column j, is the product of the i^{th} row of A (considered as a row vector) by the j^{th} column of $\bar{A}$ (considered as a column vector).

To illustrate, let

$$(20)\qquad M = \begin{bmatrix} a_{11} & a_{12} \cdots a_{1k} \\ a_{21} & a_{22} \cdots a_{2k} \\ \cdot & \cdot \quad\quad \cdot \\ \cdot & \cdot \quad\quad \cdot \\ \cdot & \cdot \quad\quad \cdot \\ a_{m1} & a_{m2} \cdots a_{mk} \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \\ \cdot \\ \cdot \\ \cdot \\ A_m \end{bmatrix}$$

$$(21)\qquad \bar{A} = \begin{bmatrix} \bar{a}_{11} & \bar{a}_{12} \cdots \bar{a}_{1n} \\ \bar{a}_{21} & \bar{a}_{22} \cdots \bar{a}_{2n} \\ \cdot & \cdot \cdot \cdot \cdot \cdot \\ \bar{a}_{k1} & \bar{a}_{k2} \cdots \bar{a}_{kn} \end{bmatrix} = [\bar{P}_1, \bar{P}_2, \ldots, \bar{P}_n]$$

where we have denoted the columns of $\bar{A}$ by $\bar{P}_j$. Then, by definition,

$$(22)\qquad A\bar{A} = \begin{bmatrix} A_1 \\ A_2 \\ \cdot \\ \cdot \\ \cdot \\ A_m \end{bmatrix} [\bar{P}_1, \bar{P}_2, \ldots, \bar{P}_n] = \begin{bmatrix} A_1\bar{P}_1 & A_1\bar{P}_2 \cdots A_1\bar{P}_n \\ A_2\bar{P}_1 & A_2\bar{P}_2 \cdots A_2\bar{P}_n \\ \cdot & \cdot \quad\quad \cdot \\ \cdot & \cdot \quad\quad \cdot \\ \cdot & \cdot \quad\quad \cdot \\ A_m\bar{P}_1 & A_m\bar{P}_2 \cdots A_m\bar{P}_n \end{bmatrix}$$

where the element in row i, column j of $A\bar{A}$ is

$$(23)\qquad A_i\bar{P}_j = a_{i1}\bar{a}_{1j} + a_{i2}\bar{a}_{2j} + \cdots + a_{ik}\bar{a}_{kj}$$

This definition is a natural generalization of the multiplication of a scalar by a row vector; for viewing $A\bar{A}$ as the multiplication of a matrix by a row of column vectors, we would expect

$$(24)\qquad A\bar{A} = A[\bar{P}_1, \bar{P}_2, \ldots, \bar{P}_n] = [A\bar{P}_1, A\bar{P}_2, \ldots, A\bar{P}_n]$$

which is clearly the case, since the j^{th} column of $A\bar{A}$ from (22) is $A\bar{P}_j$. Again, by analogy to multiplying a column vector by a scalar, we can view $A\bar{A}$ as the product of a column of row vectors by a matrix

$$(25)\qquad A\bar{A} = \begin{bmatrix} A_1 \\ A_2 \\ \cdot \\ \cdot \\ \cdot \\ A_m \end{bmatrix} \bar{A} = \begin{bmatrix} A_1\bar{A} \\ A_2\bar{A} \\ \cdot \\ \cdot \\ \cdot \\ A_m\bar{A} \end{bmatrix}$$

which again is clearly the case, since the i^{th} row of $A\bar{A}$ is $A_i\bar{A}$.

The Laws of Matrix Algebra.

Just as ordinary numbers (scalars), matrices satisfy the associative and distributive laws with respect to matrix addition and multiplication. The commutative law also holds for matrix addition. However, the commutative law with respect to matrix multiplication does not hold in general even when the matrices are square.

Let $A = [a_{jk}]$, $B = [b_{jk}]$, $C = [c_{jk}]$ each be a $J \times K$ matrix; let $D = [d_{ij}]$ be an $I \times J$ matrix; and $E = [e_{kl}]$ be a $K \times L$ matrix.

1. *The Associative Law for Addition* states:

$$(A + B) + C = A + (B + C)$$

PROOF:

$$[a_{jk} + b_{jk}] + [c_{jk}] = [a_{jk} + b_{jk} + c_{jk}] = [a_{jk}] + [b_{jk} + c_{jk}]$$

2. *The Commutative Law for Addition* states:

$$A + B = B + A$$

PROOF:

$$[a_{jk} + b_{jk}] = [b_{jk} + a_{jk}]$$

3. *The Distributive Law for Multiplication with Respect to Addition* has two forms:

$$D[A + B] = DA + DB; \quad [A + B]E = AE + BE$$

PROOF: To show the first of these, let [ ] indicate matrices, and let the summation (below) be the (i, k) element of a matrix:

$$[d_{ij}][a_{jk} + b_{jk}] = \left[\sum_{j=1}^{J} d_{ij}(a_{jk} + b_{jk})\right] = \left[\left(\sum_{j=1}^{J} d_{ij}a_{jk}\right) + \left(\sum_{j=1}^{J} d_{ij}b_{jk}\right)\right]$$

$$= \left[\sum_{j=1}^{J} d_{ij}a_{jk}\right] + \left[\sum_{j=1}^{J} d_{ij}b_{jk}\right] = DA + DB$$

4. *The Associative Law for Multiplication* states:

$$D(AE) = (DA)E$$

PROOF: Let $AE = F = [f_{jl}]$ and $DA = G = [g_{ik}]$, then

$$D(AE) = [d_{ij}][f_{jl}] = \left[\sum_{j=1}^{J} d_{ij}f_{jl}\right] = \left[\sum_{j=1}^{J} d_{ij}\left(\sum_{k=1}^{K} a_{jk}e_{kl}\right)\right];$$

$$(DA)E = [g_{ik}][e_{kl}] = \left[\sum_{k=1}^{K} g_{ik}e_{kl}\right] = \left[\sum_{k=1}^{K}\left(\sum_{j=1}^{J} d_{ij}a_{jk}\right) e_{kl}\right]$$

It will be noted that *the order of summation can be interchanged* in the last expression for $(DA)E$ and therefore the value of every (i, l) element is equal to that of $D(AE)$ shown in the equation above it.

DEFINITION: The *rank* of a matrix is the rank of the vector space generated by its columns.

THEOREM 1: *The rank of the columns of a matrix is the same as the rank of its rows.*

PROOF: Consider a homogeneous system of equations whose coefficients are the elements of the matrix. Its canonical equivalent has the same number of equations as the rank of the matrix *by rows*. Since pivot operations leave invariant any dependent or independent relation among the columns, the rank of the reduced form *by columns* is the same as the original system. But the column rank of the reduced form is the same as the row rank or the number of basic variables because the columns of these variables are unit vectors, are independent, and can be used to generate the columns by linear combinations.

8-4. INVERSE OF A MATRIX

A square $m \times m$ matrix is called *nonsingular* if the columns are *independent*. By § 8-2, Theorem 4, these m columns must form a *basis* in the space of all m-vectors because the m unit vectors form a basis. If it is possible to reduce an m-equation system to canonical form with basic variables $x_{j_1}, x_{j_2}, \ldots, x_{j_m}$, then coefficients of these variables in the *original* system viewed as vectors form a basis. To see this, note that if a set of columns is independent (or dependent) before a pivot operation, the same is true for its corresponding columns after a pivot operation and conversely. It follows that because the unit vector columns of $x_{j_1}, x_{j_2}, \ldots, x_{j_m}$ in the canonical form are obviously independent, the same is true for their correspondents in the original system.

Given any set of m independent columns of coefficients for variables $x_{j_1}, x_{j_2}, \ldots, x_{j_m}$ in an m-equation system, it is always possible to reduce the system to canonical form with these variables basic. To see this, try to reduce the system using $x_{j_1}, x_{j_2}, \ldots, x_{j_r}$. Assume, on the contrary, that it is not possible at the r^{th} stage ($r < m$) to pivot using $x_{j_{r+1}}$ (because its coefficients are all zero in the equations corresponding to the nonpivotal set). It is obvious that in this partially reduced system, column j_{r+1} can be formed as a linear combination of the r unit vectors in columns $j_1, j_2, \ldots, j_r$. But then the same is true for the corresponding columns of the original system, contradicting the independence assumption. We have therefore established:

THEOREM 1: *A set of m m-vectors is linearly independent if and only if it is possible to reduce an m-equation system to canonical form with m basic variables whose coefficients are the m-vectors.*

The above theorem provides a constructive way to determine whether or not a matrix is nonsingular. Associated with a nonsingular matrix (or basis) is another matrix known as its *inverse*, which we will illustrate below and define later. In particular, the inverse of a basis associated with the k^{th} cycle of the simplex algorithm provides a convenient way to reduce a standard linear programming problem to canonical form and provides the alternative way of performing the computations of the simplex method to be discussed in the next chapter.

An Illustration.

In system (1), x_1 and x_2 may be used for basic variables, since it can be reduced to canonical form using these variables:

(1)
$$5x_1 - 4x_2 + 13x_3 - 2x_4 + x_5 = 20 \qquad (E_1)$$
$$x_1 - x_2 + 5x_3 - x_4 + x_5 = 8 \qquad (E_2)$$

The array of coefficients of these variables is

(2)
$$\begin{bmatrix} 5 & -4 \\ 1 & -1 \end{bmatrix}$$

and, according to the above definition, constitutes the basis associated with the variables (x_1, x_2).

It is convenient to use a symbol, such as B, to denote a basis. The symbol $[a_{ij}]$ is used where the latter indicates that the element in the i^{th} row and j^{th} column of B is a_{ij}. Thus we may write, for the example above,

(3)
$$B = [a_{ij}] = \begin{bmatrix} 5 & -4 \\ 1 & -1 \end{bmatrix}$$

where $a_{11} = +5$, $a_{12} = -4$, $a_{21} = +1$, $a_{22} = -1$.

To find the inverse of the matrix (basis) B in (3), consider the canonical system of equations with basic variables y_1, y_2:

(4)
$$\begin{aligned} 5x_1 - 4x_2 + y_1 \qquad &= 0 \\ x_1 - x_2 \qquad + y_2 &= 0 \end{aligned}$$

where the coefficients of x_1 and x_2 constitute the basis B. Solve (4) for x_1 and x_2 in terms of y_1 and y_2; by elimination we obtain

(5)
$$\begin{aligned} x_1 \qquad + y_1 - 4y_2 &= 0 \\ x_2 + y_1 - 5y_2 &= 0 \end{aligned}$$

It is clear that (5) is equivalent to (4) and is in canonical form with basic variables, x_1 and x_2. The array of coefficients of y_1 and y_2 in (5) is called the *inverse* of the matrix (3) and is written B^{-1}. Hence,

(6)
$$B = \begin{bmatrix} 5 & -4 \\ 1 & -1 \end{bmatrix}; \quad B^{-1} = \begin{bmatrix} 1 & -4 \\ 1 & -5 \end{bmatrix}$$

Conversely, if the coefficients of y_1 and y_2 in (5) are considered as a matrix, then since (4) is equivalent to (5), the coefficients of x_1 and x_2 constitute the inverse of this matrix, and we immediately conclude that the inverse of the inverse of a matrix is the matrix itself. This is the analogue, for a square array of numbers, of the familiar fact that the reciprocal of the reciprocal of a number is the number itself.

The inverse of a basis may be used to reduce a linear programming system, such as (1), to canonical form relative to the associated basic variables. We interpret the first equation of (5), namely, $x_1 + y_1 - 4y_2 = 0$, to mean that if the first equation of (4) is multiplied by 1 and the second equation by -4, and the two summed, all basic variables, except x_1, will be eliminated. (If this were not so, the equating of the two different expressions would lead to a linear relation in x_1 and x_2 contradicting (4) where these variables are independent.) Similarly, from $x_2 + y_1 - 5y_2 = 0$ it follows that, if the first equation of (4) is multiplied by 1 and the second by -5, and the two summed, all basic variables, except x_2, will be eliminated. Now let us see what the effect of these same operations is on the original system, (1). Since the coefficients of x_1 and x_2 are the same as (4), these same operations performed on (1), instead of (4), will reduce (1) to canonical form with basic variables x_1 and x_2:

$$\begin{array}{lrl} (7) & x_1 \quad\;\; - \;\; 7x_3 + 2x_4 - 3x_5 = -12 & (E_1' = E_1 - 4E_2) \\ & x_2 - 12x_3 + 3x_4 - 4x_5 = -20 & (E_2' = E_1 - 5E_2) \end{array}$$

On the right in (7) are the operations required to obtain (7) from (1); note that the array of coefficients of E_1' and E_2' is the inverse of the basis given in (6).

General Properties of a Matrix and Its Inverse.

Our objective is to formalize and to prove, in general, the assertions made for the illustrative example.

A square array of numbers

$$(8) \qquad B = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{bmatrix}$$

is nonsingular and its columns constitute a *basis*, by Theorem 1, if the system of equations

$$\begin{array}{llr} (9) & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m + y_1 & = 0 \\ & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m \quad + y_2 & = 0 \\ & \cdot \quad \cdot \quad \cdot & \cdot \\ & \cdot \quad \cdot \quad \cdot & \cdot \\ & \cdot \quad \cdot \quad \cdot & \cdot \\ & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mm}x_m \qquad\qquad + y_m & = 0 \end{array}$$

is equivalent to some system (10) in canonical form, with basic variables

$(x_1, x_2, \ldots, x_m)$; i.e., B is a basis, if we can solve (9) for $x_1, x_2, \ldots, x_m$ in terms of $y_1, y_2, \ldots, y_m$ obtaining

$$
\begin{array}{lllll}
(10) & x_1 & & + \beta_{11} y_1 + \beta_{12} y_2 + \cdots + \beta_{1m} y_m = 0 \\
& & x_2 & + \beta_{21} y_1 + \beta_{22} y_2 + \cdots + \beta_{2m} y_m = 0 \\
& & \ddots & \vdots \\
& & & x_m + \beta_{m1} y_1 + \beta_{m2} y_2 + \cdots + \beta_{mm} y_m = 0
\end{array}
$$

It is clear that, if (8) is the array formed by the coefficients of some subset of m variables of an $m \times n$ linear programming problem, it is possible to reduce the problem to canonical form, using the corresponding variables as basic variables.

DEFINITION: The matrix of coefficients of y_i in (10) is the *inverse* of the matrix B of coefficients of x_j in (9). We denote the inverse of B by B^{-1}. By definition

$$
(11) \qquad B^{-1} = \begin{bmatrix} \beta_{11} & \beta_{12} \cdots \beta_{1m} \\ \beta_{21} & \beta_{22} \cdots \beta_{2m} \\ \cdot & \cdots\cdots \\ \beta_{m1} & \beta_{m2} \cdots \beta_{mm} \end{bmatrix}
$$

Theorem 1 of § 4-2 establishes the uniqueness of the canonical form, hence the uniqueness of the inverse. Conversely, since (9) is equivalent to (10) and in canonical form relative to $y_1, y_2, \ldots, y_m$, we have established both theorems that follow.

THEOREM 2: *The inverse of a basis is unique.*

THEOREM 3: *The inverse of the inverse of a matrix is the matrix itself.*

If in (10), the values of all independent variables y_i are set equal to zero, except $y_k = -1$, we obtain the obvious solution for the basic variables $x_1 = \beta_{1k}$, $x_2 = \beta_{2k}$, $\ldots$, $x_m = \beta_{mk}$. Since (9) has the same solution set, these values of x_i and y_i must also satisfy it. Substituting in the i^{th} equation of (9) yields a relation between the i^{th} row of a basis B and the k^{th} column of its inverse B^{-1}, namely,

THEOREM 4: *The sum of the products of the corresponding terms in the i^{th} row of B and k^{th} column of B^{-1} are zero or one according as $i \neq k$ or $i = k$:*

$$
(12) \qquad a_{i1}\beta_{1k} + a_{i2}\beta_{2k} + \cdots + a_{im}\beta_{mk} = \begin{cases} 0 \text{ if } i \neq k \\ 1 \text{ if } i = k \end{cases}
$$

For example, for the basis given by (2), we see that

$$
\begin{array}{llr}
(13) & a_{11}\beta_{11} + a_{12}\beta_{21} = \quad (5)(1) + \quad (-4)(1) = 1 & (i = 1, k = 1) \\
& a_{11}\beta_{12} + a_{12}\beta_{22} = (5)(-4) + (-4)(-5) = 0 & (i = 1, k = 2) \\
& a_{21}\beta_{11} + a_{22}\beta_{21} = \quad (1)(1) + \quad (-1)(1) = 0 & (i = 2, k = 1) \\
& a_{21}\beta_{12} + a_{22}\beta_{22} = (1)(-4) + (-1)(-5) = 1 & (i = 2, k = 2)
\end{array}
$$

Having established Theorem 4, we may proceed to interchange the roles of B and B^{-1} to obtain

THEOREM 5: *The sum of the products of corresponding terms in the* i^{th} *row of* B^{-1} *and* k^{th} *column of* B *are zero or one, according as* $i \neq k$ *or* $i = k$:

$$\beta_{i1}a_{1k} + \beta_{i2}a_{2k} + \cdots + \beta_{im}a_{mk} = \begin{cases} 0 \text{ if } k \neq i \\ 1 \text{ if } k = i \end{cases} \tag{14}$$

For our example, we observe that

$$\begin{aligned} \beta_{11}a_{11} + \beta_{12}a_{21} &= (1)(5) + (-4)(1) = 1 \quad (i = 1, k = 1) \\ \beta_{11}a_{12} + \beta_{12}a_{22} &= (1)(-4) + (-4)(-1) = 0 \quad (i = 1, k = 2) \\ \beta_{21}a_{11} + \beta_{22}a_{21} &= (1)(5) + (-5)(1) = 0 \quad (i = 2, k = 1) \\ \beta_{21}a_{12} + \beta_{22}a_{22} &= (1)(-4) + (-5)(-1) = 1 \quad (i = 2, k = 2) \end{aligned} \tag{15}$$

THEOREM 6: *If a canonical system* (10) *can be formed from a canonical system* (9) *by linear combinations, it is equivalent to* (9), *and the array of coefficients of the* y_i *in* (10) *is the inverse of the basis, and conversely.*

PROOF: Consider the combined system (9) and (10). By § 8-1, Theorems 8, 9, 10, the rank of the system is m because the first m equations are independent and by hypothesis the remaining m are dependent upon them. However, the last m equations are independent and, since m is the maximum number that can be independent, this implies the first m equations of (9) are dependent on (10). Hence, (10) implies (9) and the two systems are equivalent. The rest of the theorem follows by the definition of the inverse.

Let us now consider another theorem, the *converse* of Theorem 4 (or of Theorem 5). Suppose we are given system (10) with an array of coefficients $[\beta_{ij}]$ and another array of coefficients $[a_{ij}]$ which satisfy the row-column relations (12). We wish to prove that (9) is equivalent to (10) and hence $[a_{ij}]$ is the inverse of $[\beta_{ij}]$.

To see this, multiply the first equation of (10) by a_{i1}, the second by $a_{i2}, \ldots$, the m^{th} equation by a_{im}, and sum; we will obtain the i^{th} relation of (9). Thus (12) and (10) imply (9). Applying Theorem 6, we have shown

THEOREM 7: *A necessary and sufficient condition that the inverse of* $[a_{ij}]$ *is* $[\beta_{ij}]$ *is that the row-column relations* (12) *or* (14) *hold.*

Recall that the *transpose* of a basis B is an $m \times m$ array of elements obtained by interchanging rows and columns of B; it is left as an exercise to prove that relations (12) and (14) imply:

THEOREM 8: *The inverse of the transpose of a basis is the transpose of the inverse of a basis.*

The basis B consisting of all ones down the main diagonal and zero elsewhere is called the *identity matrix* and is given the symbol I or I_m: it is so called because for any $m \times n$ matrix M, $I_mM = M$. For example, the identity matrix for $m = 4$ is

$$B = I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (m = 4) \tag{16}$$

It is easy to verify that $B^{-1} = B$ because the associated system (9) is in this case in canonical form with respect to both x_1, x_2, x_3, x_4 and y_1, y_2, y_3, y_4.

Reduction of an $m \times n$ System to Canonical Form.

To reduce

$$\begin{array}{c} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{array} \tag{17}$$

to canonical form with basic variables $x_1, x_2, \dots, x_m$, assume that the square array $B = [a_{ij}]$ is a basis and its inverse $B^{-1} = [\beta_{ij}]$ is known. If the first equation of (17) is multiplied by β_{11}, the second by $\beta_{12}, \dots$, the m^{th} by β_{1m}, then the weighted sum is

$$\left(\sum_{k=1}^{m} \beta_{1k}a_{k1}\right) x_1 + \left(\sum_{k=1}^{m} \beta_{1k}a_{k2}\right) x_2 + \dots + \left(\sum_{k=1}^{m} \beta_{1k}a_{kn}\right) x_n = \sum_{k=1}^{m} \beta_{1k}b_k \tag{18}$$

In general, the r^{th} equation of system (19) can be generated by multiplying the first equation of (17) by β_{r1}, the second by $\beta_{r2}, \dots$, the m^{th} by β_{rm} and forming the weighted sum; this will result in the canonical system:

$$\begin{array}{lcl} x_1 & + \bar{a}_{1,m+1}x_{m+1} + \dots + \bar{a}_{1n}x_n = & \bar{b}_1 \\ \quad\vdots & \vdots & \vdots \\ \qquad x_r & + \bar{a}_{r,m+1}x_{m+1} + \dots + \bar{a}_{rn}x_n = & \bar{b}_r \\ \qquad\quad\vdots & \vdots & \vdots \\ \qquad\qquad x_m & + \bar{a}_{m,m+1}x_{m+1} + \dots + \bar{a}_{mn}x_n = & \bar{b}_m \end{array} \tag{19}$$

where, for $j = 1, 2, \dots, n$, we have set

$$\begin{array}{c} \bar{a}_{1j} = \beta_{11}a_{1j} + \beta_{12}a_{2j} + \dots + \beta_{1m}a_{mj} \\ \bar{a}_{2j} = \beta_{21}a_{1j} + \beta_{22}a_{2j} + \dots + \beta_{2m}a_{mj} \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \bar{a}_{mj} = \beta_{m1}a_{1j} + \beta_{m2}a_{2j} + \dots + \beta_{mm}a_{mj} \end{array} \tag{20}$$

and

$$\begin{array}{c} \bar{b}_1 = \beta_{11}b_1 + \beta_{12}b_2 + \dots + \beta_{1m}b_m \\ \bar{b}_2 = \beta_{21}b_1 + \beta_{22}b_2 + \dots + \beta_{2m}b_m \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \bar{b}_m = \beta_{m1}b_1 + \beta_{m2}b_2 + \dots + \beta_{mm}b_m \end{array} \tag{21}$$

Note that (19) is in *canonical* form with respect to $x_1, x_2, \ldots, x_m$ because of the row-column relationship between B^{-1} and B; namely by (14), it follows for $j = 1, 2, \ldots, m$, that

$$\bar{a}_{ij} = \begin{cases} 0 & \text{for } i = 1, 2, \ldots, m \text{ and } i \neq j \\ 1 & \text{for } i = j \end{cases} \tag{22}$$

8-5. THE SIMPLEX ALGORITHM IN MATRIX FORM

The central problem in vector notation is to find $x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0$ and Min z, satisfying

$$P_1x_1 + P_2x_2 + \ldots + P_nx_n = Q \tag{1}$$

$$c_1x_1 + c_2x_2 + \ldots + c_nx_n = z \tag{2}$$

where

$$P_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \cdot \\ \cdot \\ \cdot \\ a_{mj} \end{bmatrix}; \quad Q = \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_m \end{bmatrix} \tag{3}$$

and a_{ij}, b_i, c_j are constants.

It is required for the simplex algorithm that m of the vectors P_j be independent.[2] Let $P_{j_1}, P_{j_2}, \ldots, P_{j_m}$ be such a set of independent vectors. These form a *basis*, B, in the vector space generated by $P_1, P_2, \ldots, P_n$:

$$B = [P_{j_1}, P_{j_2}, \ldots, P_{j_m}] \tag{4}$$

A *canonical* form is obtained by multiplying (1) by B^{-1}, i.e.,

$$(B^{-1}P_1)x_1 + (B^{-1}P_2)x_2 + \ldots + (B^{-1}P_n)x_n = B^{-1}Q \tag{5}$$

or

$$\bar{P}_1x_1 + \bar{P}_2x_2 + \ldots + \bar{P}_nx_n = \bar{Q} \tag{6}$$

where (see § 8-2)

$$B^{-1}P_j = \bar{P}_j; \; B^{-1}Q = \bar{Q} \tag{7}$$

are the *representations* of P_j and Q, respectively in terms of the basis. Note that from $B^{-1}B = I$ (identity matrix) follows

$$\bar{P}_{j_i} = B^{-1}P_{j_i} = U_i \tag{8}$$

where U_i is a unit vector with unity in component i and zero elsewhere. But the latter, by definition, means (6) is in canonical form with "basic" variables $x_{j_1}, x_{j_2}, \ldots, x_{j_m}$. (See § 4-2.)

[2] Phase I of the simplex algorithm takes care of the situation where this is not true.

The basic solution is obtained by setting non-basic variables $x_j = 0$; thus the values of the basic variables are given by

(9) $$U_1 x_{j_1} + U_2 x_{j_2} + \ldots + U_m x_{j_m} = \bar{Q}$$

or

(10) $$\begin{bmatrix} x_{j_1} \\ x_{j_2} \\ \cdot \\ \cdot \\ \cdot \\ x_{j_m} \end{bmatrix} = \bar{Q}$$

EXERCISE: Show what would be affected by a change in the *ordering* of the basic variables and the basis vectors.

The basic solution is *feasible*, if

(11) $$\bar{Q} \geq 0$$

DEFINITION: $\bar{Q} \geq 0$ means each component $\bar{b}_i$ of $\bar{Q}$ satisfies $\bar{b}_i \geq 0$.

The relative cost factors, $\bar{c}_j$, are obtained by eliminating x_{j_i} from the z-equation. If we define the row vector

(12) $$\gamma = [c_{j_1}, c_{j_2}, \ldots, c_{j_m}]$$

and multiply (6) by γ, we obtain

(13) $$(\gamma\bar{P}_1)x_1 + (\gamma\bar{P}_2)x_2 + \ldots + (\gamma\bar{P}_n)x_n = (\gamma\bar{Q})$$

where $(\gamma\bar{P}_j)$ are *constants* (for each is the product of a row vector by a column vector). In particular, $\gamma\bar{P}_{j_i} = \gamma U_i = c_{j_i}$, so that (13) has the same coefficients for the basic variables as does (2). Hence, by subtracting (13) from (2), we eliminate the basic variables, obtaining

(14) $$(c_1 - \gamma\bar{P}_1)x_1 + (c_2 - \gamma\bar{P}_2)x_2 + \ldots + (c_n - \gamma\bar{P}_n)x_n = z - \gamma\bar{Q}$$

Therefore the relative cost factors are given by

(15) $$\begin{aligned} \bar{c}_j &= c_j - \gamma\bar{P}_j \\ &= c_j - \gamma(B^{-1}P_j) \\ &= c_j - (\gamma B^{-1})P_j \end{aligned}$$

or

(16) $$\bar{c}_j = c_j - \pi P_j$$

where we have set the row vector

(17) $$\pi = \gamma B^{-1}$$

In words, (16) states that the relative cost coefficients, $\bar{c}_j$, are obtained by subtracting from c_j a weighted sum of the coefficients $a_{1j}, a_{2j}, \ldots, a_{mj}$, where the weights (the same for all j) are the m components $\pi_1, \pi_2, \ldots, \pi_m$

of π. The elements π_i are called *simplex multipliers* (these will be discussed more fully in the next chapter). Multiplying (17) by B, we obtain

$$\pi B = \gamma \tag{18}$$

or

$$\pi(P_{j_1}, P_{j_2}, \ldots, P_{j_m}) = (c_{j_1}, c_{j_2}, \ldots, c_{j_m}) \tag{19}$$

Hence, in particular,

$$\pi P_{j_i} = c_{j_i} \qquad \text{for } i = 1, 2, \ldots, m \tag{20}$$

Thus the weights π_i are just the numbers required to multiply through the *original* equations (1) and sum in order to eliminate the coefficients of the basic variables from (2).

The basic solution is optimal, if all $\bar{c}_j \geq 0$. If not all $\bar{c}_j \geq 0$, then an improved solution is sought by first choosing s, such that

$$\bar{c}_s = \text{Min } \bar{c}_j \tag{21}$$

and then increasing the value of x_s as much as possible, keeping other non-basic variables at zero. In order to be nonnegative, the vector of values of the basic variables must satisfy

$$(\bar{Q} - \bar{P}_s x_s) \geq 0 \tag{22}$$

At some critical value $x_s = x_s^*$, the value of some component r of this vector will change sign while all others remain nonnegative (otherwise $z \to -\infty$ as $x_s \to +\infty$). The components of $\bar{Q}$, $\bar{P}_s$, and r are defined by our earlier notation to be

$$\bar{Q} = \begin{bmatrix} \bar{b}_1 \\ \bar{b}_2 \\ \cdot \\ \cdot \\ \cdot \\ \bar{b}_m \end{bmatrix}; \quad \bar{P}_s = \begin{bmatrix} \bar{a}_{1s} \\ \bar{a}_{2s} \\ \cdot \\ \cdot \\ \cdot \\ \bar{a}_{ms} \end{bmatrix}; \quad x_s^* = \frac{\bar{b}_r}{\bar{a}_{rs}} = \underset{\bar{a}_{is}>0}{\text{Min}} \frac{\bar{b}_i}{\bar{a}_{is}} \qquad (\bar{a}_{rs} > 0) \tag{23}$$

Hence, P_{j_r} is replaced in the basis by P_s to form the basis B^* of the next cycle. This completes the description of the simplex process in matrix notation. We shall now go deeper into the nature of the transformations from cycle to cycle.

The Transformations from Cycle k to $k + 1$.

The last step of the simplex process is to transform the tableau by pivoting on $\bar{a}_{rs}$. Instead, here we shall use the inverse of the new basis to adjust slightly the representations of P_j and Q in terms of the *old* basis,

B, given by (7) to obtain their representations in terms of the *new* basis, B^*. First, we note that $\bar{P}_j = B^{-1}P_j$ or $P_j = B\bar{P}_j$, so that

(24)
$$P_s = [P_{j_1}, P_{j_2}, \ldots, P_{j_m}] \begin{bmatrix} \bar{a}_{1s} \\ \bar{a}_{2s} \\ \cdot \\ \cdot \\ \cdot \\ \bar{a}_{ms} \end{bmatrix} = P_{j_1}\bar{a}_{1s} + \ldots + P_{j_r}\bar{a}_{rs} + \ldots + P_{j_m}\bar{a}_{ms}$$

where $\langle \bar{a}_{1s}, \bar{a}_{2s}, \ldots, \bar{a}_{ms} \rangle$ is the representation of P_s in terms of B. We may use (24) to express P_{j_r} in terms of the new basis B^*; thus

(25) $$P_{j_r} = P_{j_1}k_1 + \ldots + P_s k_r + \ldots + P_{j_m}k_m = B^*K$$

where we have set $K = \{k_1, k_2, \ldots, k_m\}$, $B^* = [P_{j_1}, \ldots, P_s, \ldots, P_{j_m}]$ and

(26) $$k_i = -\bar{a}_{is}/\bar{a}_{rs} \qquad (i \neq r)$$

(27) $$k_r = 1/\bar{a}_{rs}$$

For all other $i \neq r$ we may trivially represent P_{j_i} in terms of B^*,

(28) $$P_{j_i} = P_{j_1} \cdot 0 + \ldots + P_s \cdot 0 + \ldots + P_{j_i} \cdot 1 + \ldots + P_{j_m} \cdot 0 = B^*U_i$$

so that *the relation between the old and new basis is given by*

(29) $$B = [P_{j_1}, P_{j_2}, \ldots, P_{j_m}] = B^*[U_1, U_2, \ldots, K, \ldots, U_m]$$

Multiplying through on the right by B^{-1} and by $(B^*)^{-1}$ on the left, we obtain *the relation between the inverse of the new basis and the previous inverse:*

(30) $$(B^*)^{-1} = [U_1, U_2, \ldots, K, \ldots, U_m]B^{-1}$$

Matrix (31) is practically the identity matrix, except that column r consists of k_i values. A matrix that differs from the identity in just one row (or column) is called an *elementary matrix.*

(31)
$$[U_1, U_2, \ldots, K, \ldots, U_m] = \begin{bmatrix} 1 & & & k_1 & & & \\ & 1 & & \cdot & & & \\ & & \cdot & \cdot & & & \\ & & & \cdot\;\cdot & & & \\ & & & k_r & & & \\ & & & \cdot\;\cdot & & & \\ & & & \cdot & \cdot & & \\ & & & \cdot & & 1 & \\ & & & k_m & & & 1 \end{bmatrix}$$

Thus, according to (30), *the new inverse is the product of an elementary matrix and the inverse of the previous basis.* If we now multiply both sides of (30) on the right by any P_j, we can obtain the representation of P_j in terms of the new basis from its representation in terms of the old basis, $\bar{P}_j = B^{-1}P_j$:

(32) $$(B^*)^{-1}P_j = [U_1, U_2, \ldots, K, \ldots, U_m]\bar{P}_j$$

It is convenient to write matrix (31) as the *sum* of an identity matrix and a null matrix except for one column:

(33) $$[U_1, U_2, \ldots, K, \ldots, U_m] = [U_1, U_2, \ldots, U_r, \ldots, U_m] + [0, 0, \ldots, K - U_r, \ldots, 0]$$

and to write the vector

(34) $$\bar{K} = K - U_r = \begin{bmatrix} \bar{k}_1 \\ \cdot \\ \cdot \\ \cdot \\ \bar{k}_r \\ \cdot \\ \cdot \\ \cdot \\ \bar{k}_m \end{bmatrix} = \begin{bmatrix} k_1 \\ \cdot \\ \cdot \\ \cdot \\ k_r - 1 \\ \cdot \\ \cdot \\ \cdot \\ k_m \end{bmatrix}$$

We now have

(35) $$\begin{aligned}(B^*)^{-1} &= [U_1, U_2, \ldots, U_r, \ldots, U_m]B^{-1} + [0, 0, \ldots, K - U_r, 0, \ldots, 0]B^{-1} \\ &= B^{-1} + [0, 0, \ldots, \bar{K}, \ldots, 0]B^{-1}\end{aligned}$$

If now we denote the *rows* of B^{-1} by β_i, so that

(36) $$B^{-1} = \begin{bmatrix} \beta_1 \\ \cdot \\ \cdot \\ \cdot \\ \beta_r \\ \cdot \\ \cdot \\ \cdot \\ \beta_m \end{bmatrix}$$

and substitute above, we have

(37) $$(B^*)^{-1} = B^{-1} + [0, 0, \ldots, \bar{K}, \ldots, 0] \begin{bmatrix} \beta_1 \\ \cdot \\ \cdot \\ \cdot \\ \beta_r \\ \cdot \\ \cdot \\ \cdot \\ \beta_m \end{bmatrix}$$

(38) $$= B^{-1} + \bar{K}\beta_r$$

Note that $(B^*)^{-1}$ *differs from* B^{-1} *by a matrix* $\bar{K}\beta_r$, *which is the product of a column vector* $\bar{K}$ *and a row vector* β_r. Thus

$$(39) \qquad \bar{K}\beta_r = \begin{bmatrix} \bar{k}_1 \\ \cdot \\ \cdot \\ \cdot \\ \bar{k}_i \\ \cdot \\ \cdot \\ \cdot \\ \bar{k}_m \end{bmatrix} [\beta_{r1}, \ldots, \beta_{rj}, \ldots, \beta_{rm}], \quad B^{-1} = [\beta_{ij}]$$

The (i, j) element of $\bar{K}\beta_r$ is simply $\bar{k}_i\beta_{rj}$. Hence, to form the (i, j) element of $(B^*)^{-1}$, we add $\bar{k}_i\beta_{rj}$ to β_{ij}; i.e.,

$$(40) \qquad [B^*]^{-1} = [\beta_{ij}] + [\bar{k}_i\beta_{rj}]$$

Finally, to form the new representation from the old we have from (38)

$$(41) \qquad \begin{aligned} (B^*)^{-1}P_j &= (B^{-1} + \bar{K}\beta_r)P_j = B^{-1}P_j + (\bar{K}\beta_r)P_j \\ &= \bar{P}_j + \bar{K}\bar{a}_{rj} \end{aligned}$$

where we have replaced the constant $\beta_r P_j$ by $\bar{a}_{rj}$, the value of the r^{th} component in the representation of P_j in terms of B. *Thus, the new* $\bar{P}_j$ *differs from the old by a vector proportional to* $\bar{K}$; *the factor of proportionality is the* r^{th} *component of* $\bar{P}_j$.

Product Form of the Inverse.

Relations (30) and (40) are two ways to express the new inverse in terms of the old. It will be noted that (40) requires in general m^2 changes in the components of B^{-1}; whereas (30) shows that the *process* of obtaining $(B^*)^{-1}$ from B^{-1}, by multiplying by the elementary matrix defined by (31), requires only knowledge of the m components of the vector K and its column location r in the matrix.

A. Orden, in the early days of linear programming, proposed that it can be computationally convenient to represent the inverse of the basis as a product of elementary matrices. For example, the inverse of the *initial* basis could always be arranged to be the *identity* by using artificial variables. The inverse of the basis for cycle 1 would then be a single elementary matrix which could be easily recorded on a magnetic tape of an electronic computer as the single vector column K (and its location r). The inverse of the basis for cycle 2 would then be the product of a new elementary matrix and the previous one for cycle 1. This product could be stored by simply recording the new column K *after* the first column K on the same magnetic tape, etc. Both the Orchard-Hays–RAND Code [1956-1] and the Philip Wolfe–RAND Code (using a flexible language medium for the IBM-704 Computer) make use of Orden's suggestion for recording the inverse.

EXERCISE: Review the relationship between the vector K in (31) and the representation of the new vector P_s entering the basis.

EXERCISE: Suppose the inverse of the basis is given in product form; determine the detailed computational process of representing a vector P_s in terms of a basis by multiplying it on the left by the successive elementary matrices generated by cycle 1, cycle 2, etc.

Block-Pivoting.

Tucker [1960-3] generalizes the notion of pivot by introducing several columns into the basic set at once. With regard to the detached coefficient array (42), let $x_{m+1}, x_{m+2}, \ldots, x_{m+k}$ replace $x_1, x_2, \ldots, x_k$ as basic variables.

$$(42)\quad A = \left[\begin{array}{cccccccc} 1 & & & & \bar{a}_{1m+1} \cdots \bar{a}_{1m+k} & \cdots & \bar{a}_{1n} & \bar{b}_1 \\ & \ddots & & & & & & \vdots \\ & & 1 & & \bar{a}_{km+1} \cdots \bar{a}_{km+k} & & & \vdots \\ & & & 1 & & & & \vdots \\ & & & & \ddots & & & \vdots \\ & & & & 1 \quad \bar{a}_{mm+1} \cdots \bar{a}_{mm+k} & \cdots & \bar{a}_{mn} & \bar{b}_m \end{array}\right]$$

←— basis —→|← entering the basis →|← other columns →| constants |

Note that the new basis has the structure

$$B^* = \begin{bmatrix} P & 0 \\ Q & I_{m-k} \end{bmatrix}$$

where P represents the square block array dotted in (42) called the *block-pivot*. Since the value of the determinant of B^* is the same as the value of the determinant of P, it follows that in order for B^* to be a basis it is necessary that the determinant of P be non-zero. To "pivot," let P^{-1} be the inverse of P. Analogous to the first step of ordinary pivoting (of dividing through by the non-zero pivot coefficient) the first k rows of (42) are multiplied by P^{-1}. Let the original array in matrix form be

$$(43)\qquad A = \begin{bmatrix} I_k & 0 & P & R & e \\ 0 & I_{m-k} & Q & S & f \end{bmatrix}$$

Then multiplying by P^{-1} yields

$$(44)\qquad A' = \begin{bmatrix} P^{-1} & 0 & I_k & P^{-1}R & P^{-1}e \\ 0 & I_{m-k} & Q & S & f \end{bmatrix}$$

The next step is to "eliminate" the set of variables $x_{m+1}, \ldots, x_{m+k}$ from the remaining equations. To do this, the first k rows are multiplied by $-Q$ on the left and added to the bottom rows, yielding the new array

$$A^* = \begin{bmatrix} P^{-1} & 0 & I_k & P^{-1}R & P^{-1}e \\ -QP^{-1} & I_{m-k} & 0 & S - QP^{-1}R & f - QP^{-1}e \end{bmatrix}$$

Note that the columns corresponding to the new basis when properly ordered are an identity matrix so that A^* is in required canonical form.

8-6. PROBLEMS

Review.

1. Prove the values of $\bar{a}_{ij}$ in the canonical form do not depend, in general, on the order of elimination provided only that the unit coefficient of each basic variable in the canonical system is in the same row. If not, the canonical forms will be identical after proper reordering of the rows.
2. For the following, determine if each system is consistent or inconsistent, and if there are any redundant equations. If consistent, determine its rank.

(a)
$$\begin{aligned} 2x_1 - 2x_2 + x_3 &= 3 \\ 2x_1 + x_2 - 2x_3 &= 2 \\ 5x_1 + x_2 + x_3 &= 3 \\ x_2 - x_3 &= 1 \end{aligned}$$

(b)
$$\begin{aligned} 2x_1 - x_2 + 3x_3 &= 1 \\ -4x_1 + 3x_2 + x_3 &= 3 \\ -5x_1 + 4x_2 + 3x_3 &= 5 \\ x_1 + 2x_2 + x_3 &= 2 \end{aligned}$$

(c)
$$\begin{aligned} x_1 + x_2 + 3x_3 + x_4 + x_5 + 6x_6 &= 1 \\ 2x_1 - x_2 - x_3 + x_4 - 2x_5 - 2x_6 &= 3 \\ 4x_1 + x_2 + 5x_3 + 3x_4 \qquad + 10x_6 &= 5 \\ 6x_1 - x_2 - 9x_3 + 2x_4 - 7x_5 + 12x_6 &= 5 \end{aligned}$$

3. The classical Hitchcock-Koopmans transportation problem consists in finding nonnegative solutions to the system

$$\sum_{j=1}^{n} x_{ij} = a_i \qquad (i = 1, 2, \ldots, m;\ a_i \geq 0)$$

$$\sum_{i=1}^{m} x_{ij} = b_j \qquad (j = 1, 2, \ldots, n;\ b_j \geq 0)$$

$$\sum_{i=1}^{m} \sum_{j=1}^{n} d_{ij}x_{ij} = z \text{ (Min)}$$

Show that $\sum_1^m a_i = \sum_1^n b_j$ is necessary for the equations to be consistent.

4. What is the rank of a transportation problem without slacks? With slacks? Give proof (see Chapter 3, Problems 4 and 5).
5. Given two linear systems, how would you show whether or not they have the same solution set? Are equivalent? Prove that system (A) and (B) are equivalent.

$$\text{(A)}\quad \begin{aligned} 2x_1 + 3x_2 + 4x_3 &= 9 \\ x_1 - x_2 + x_3 &= 1 \\ 4x_1 + 3x_2 + 2x_3 &= 9 \end{aligned} \qquad \text{(B)}\quad \begin{aligned} x_1 + x_2 + x_3 &= 3 \\ 7x_1 - 2x_2 + 5x_3 &= 10 \\ 5x_1 - 2x_2 + 7x_3 &= 10 \end{aligned}$$

6. For solvable systems of rank r, show that there is only one way to form a dependent $(r + 1)^{st}$ equation as a linear combination of r independent equations.
7. Given a set of r independent equations and a set of $m - r$ dependent equations, prove that the role of any independent equation and any dependent equation can be interchanged providing there is a non-zero weight on the independent equation in forming the dependent equation as a linear combination of the independent equations.

Invariance Properties under Pivoting. (Refer to § 8-1.)

8. Construct an example to show that a sequence of elementary operations does not preserve one-to-one correspondence of solvable independent equations and of the remaining dependent or contradictory equations as does a sequence of pivot operations.
9. Find the rank r of the system below by finding the number of equations in the canonical equivalent. Find the largest number of independent equations of the original system and check if this number is equal to the rank. Show that this is the same as the rank of the matrix of coefficients and constant terms.

$$\begin{aligned} 2x_1 + 3x_2 + 4x_3 &= 9 \\ x_1 - x_2 + x_3 &= 1 \\ 3x_1 + 2x_2 + 5x_3 &= 10 \\ 4x_1 + x_2 + 6x_3 &= 11 \\ 6x_1 + 4x_2 + 10x_3 &= 20 \end{aligned}$$

Show how to generate all solutions to this system of equations.

10. How is the largest number of independent equations of a system generated? How does one determine whether a system is consistent or inconsistent? Does an inconsistent system have rank? Show that if the rank of the matrix of coefficients and constant terms is the same after deletion of the constant terms, the system is solvable.
11. Why does any set of independent equations equivalent to a given solvable system have the same number of equations as the rank of the system?

12. If a given system has a set of k independent equations and the remaining equations are dependent upon them, show that k is the maximum number of independent equations in the system.
13. Show that systems generated by successive elementary transformations from a given system have the same rank.
14. Let $x_1 = x_1^o, \ldots, x_k = x_k^o$ and $x_{k+1} = \ldots = x_n = 0$ be a solution to a system of equations where $x_i^o \neq 0$ for $i = 1, 2, \ldots, k$. Suppose r is the rank of the subsystem formed by dropping terms in $x_{k+1}, \ldots, x_n$. Show there exists a solution involving no more than r variables with non-zero values.
15. Suppose no upper bound on the objective function z for a system of linear equations in nonnegative variables exists; let k be the minimum number of positive variables necessary to achieve a class of solutions in which $z \to +\infty$. Show that $k = r + 1$ where r is the rank of the subsystem formed by dropping all variables of zero value in the above solution.
16. Suppose $\sum_i x_{ijk} = a_{jk}$, $\sum_j x_{ijk} = b_{ik}$, $\sum_k x_{ijk} = c_{ij}$, where $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$; $k = 1, 2, \ldots, p$. What relations must be satisfied by the a_{jk}, b_{ik}, and c_{ij} for the system to be consistent? How many equations are independent?

Vector Spaces. (Refer to § 8-2.)

17. Review the definition of an independent set of vectors; show that a single vector is an independent vector, except the null vector. Show also that the null vector is not part of any independent set.
    (a) Show that if $P_1, P_2, \ldots, P_n$ and Q are m-component column vectors and

$$P_1x_1 + P_2x_2 + \ldots + P_nx_n = Q$$

    where the x_j are scalars, then for any scalar k,

$$P_1(kx_1) + P_2(kx_2) + \ldots + P_n(kx_n) = Qk$$

    (b) Show that if $P_1y_1 + P_2y_2 + \ldots + P_ny_n = R$ also holds, then

$$P_1(y_1 + kx_1) + P_2(y_2 + kx_2) + \ldots + P_n(y_n + kx_n) = Q + kR$$

18. Show that if a system of linear equations is written in vector form

(a) $$P_1x_1 + P_2x_2 + \ldots + P_nx_n = Q$$

where P_j and Q are the j^{th} column vector of coefficients and constant terms respectively, then

(b) $$P_1'x_1 + P_2'x_2 + \ldots + P_n'x_n = Q'$$

where P_j' and Q' are the corresponding columns after an elementary transformation.

19. Show in Problem 18 that if $P_1, P_2, \ldots, P_k$ are linearly independent, then $P_1', P_2', \ldots, P_k'$ are also and if there is a linear dependence relation between $P_1, P_2, \ldots, P_k$, the same relation holds for $P_1', P_2', \ldots, P_k'$.

Matrices. (Refer to § 8-3.)

Let $A_1 = [4, \quad 4, \quad 2]$ $\qquad P_1 = \{1, \quad -2, \quad 3\}$

$A_2 = [6, \quad 3, \quad -1]$

20. Show that $A_1P_1 = 2$ and that $A_2P_1 = -3$.
21. Find $3A_2$; $A_1 + A_2$; $A_1 + 3A_2$.
22. If $A_1 + A_3 = A_2$, what are the components of A_3?
23. Suppose $A_1 = [2, 1]$, $A_2 = [1, -1]$, and $R = \{x_1, x_2\}$. If $A_1P_1 = 1$ and $A_2P_1 = 3$, what are the components of P_1?
24. A buyer for a department store bought 10 dresses at \$12.00 each, 15 sweaters at \$6.00 each, 3 suits at \$40.00 each, and 20 blouses at \$4.00 each. Let the vector $A = [10, 15, 3, 20]$ represent the quantities and $P = \{12, 6, 40, 4\}$ the price vector. Show by vector multiplication that the total value of his purchases is \$370.
25. A plastics manufacturer discovers that the molding machine set-up time for molding a certain part requires two men for three hours. The pay scale is \$20.00 per hour for set-up men. Suppose each part requires 20 seconds for molding. Labor costs, including overhead, are \$2.50 per hour. Also the part requires 2 ounces of material which costs \$.16 per pound. Write a four component row vector that represents the costs of producing one part, each of two parts, each of three parts, etc. Using vector multiplication, find the cost of producing one part. By vector operations find the total cost of a run of 300 parts.
26. Find the components of $X = \{x_1, x_2\}$ where

$$\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

27. If A is a row vector and P a column vector, show that $A(kP) = k(AP)$, where k is a constant.
28. Perform the indicated operations:

(a) $\begin{bmatrix} 2 & 3 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 3 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 3 \end{bmatrix}$

(d) $[2 \quad -1] \begin{bmatrix} 2 & 3 \\ 1 & 3 \end{bmatrix}$

(e) $\begin{bmatrix} 2 & 3 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 4 & 1 & 3 & -1 \\ 2 & 0 & 2 & 2 \end{bmatrix}$

(f) $3 \begin{bmatrix} 2 & 3 \\ 1 & 3 \end{bmatrix} - 2 \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

29. Let I be a 3×3 identity matrix and M any 3×3 matrix. Show that $MI = IM = M$.
30. Let O be a square null matrix (all elements zero). Show that $MO = OM = O$.
31. Let $M = \begin{bmatrix} 3 & 1 & 2 \\ -1 & 0 & 2 \\ 1 & 2 & 1 \end{bmatrix}$, and I and O be defined as in Problems 29 and 30. Find

(a) M^2, M^3, M^4

(b) I^2, I^3, I^4

(c) O^2, O^3, O^4

Inverse of a Matrix. (Refer to § 8-4.)

32. Find the inverse of each of the following matrices:

(a) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

(b) $\begin{bmatrix} 3 & 2 & 1 \\ 1 & -2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$

33. What are the inverses of each of the bases of examples 1 and 2, § 5-2? For each inverse show that relations (12) and (14) of § 8-4 hold.
34. Each element B_{ij} of the inverse B^{-1} of B can be written as $D_{ji}(-1)^{i+j}/D$, where D is the determinant associated with B, and D_{ji} is the determinant formed by dropping row i and column j of B. Show that this is true.
35. The familiar equations for the rotation of coordinates are given by

$$y_1 = x_1 \cos\theta - x_2 \sin\theta$$
$$y_2 = x_1 \sin\theta + x_2 \cos\theta$$

Solve for x_1 and x_2 in terms of y_1 and y_2. What is the inverse of the basis? Show that relations (12) and (14) of § 8-4 hold.

36. (a) Find the inverse of the coefficients of x_1 and x_2 in

$$\begin{aligned} 3x_1 - 2x_2 + 4x_3 + 2x_4 - x_5 + x_6 \quad &= 4 \\ x_1 + x_2 + x_3 + 3x_4 + x_5 \quad + x_7 &= 3 \end{aligned}$$

(b) Reduce to canonical form relative to x_1 and x_2.

How do the coefficients of x_6 and x_7 compare with the elements of the inverse?

37. Show in general that the elements of the inverse of any set of basic variables of the $m \times n$ system ($m \leq n$) of nonnegative variables

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + x_{n+1} \qquad\qquad &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \quad + x_{n+2} \qquad &= b_2 \\
\vdots \qquad\qquad\qquad\qquad\qquad\qquad & \quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \qquad\qquad + x_{m+n} &= b_m
\end{aligned}$$

will be the coefficients of $x_{n+1}, x_{n+2}, \ldots, x_{n+m}$ when the system is reduced to canonical form.

38. Show that if $x_1, x_2, \ldots, x_m$ is a basic set of variables (so that it is possible to reduce Problem 17 to canonical form relative to these variables by a series of elementary operations) that $P_1, P_2, \ldots, P_m$ are linearly independent and form a basis in m-dimensional coordinate space.

39. Show that the rank of a matrix is the same as the rank of the vector space generated by its row vectors. Compare with the definition given in § 8-2.

40. Show that the determinant of an $m \times m$ matrix vanishes if its rank r is less than m and does not vanish if its rank is m.

41. (a) Given $\sum_{j=1}^{n} a_{ij}x_j = y_i$ for $i = 1, 2, \ldots, m$ (see § 8-2-(2)), show that particular values of a_{ij} and y_i can be chosen so that
    (i) there is *no* set of values of x_j that satisfy the system;
    (ii) there is a *unique* set of values of x_j that satisfy the system;
    (iii) there are *many* sets of values of x_j that satisfy the system.

    (b) Prove: If there is always a unique set of x_j satisfying the system whatever be the choice of $y_1, y_2, \ldots, y_m$, then $n = m$ and $[a_{ij}]$ is a basis.

The Simplex Method in Matrix Form. (Refer to § 8-5.)

42. Show that if $P_1, P_2, \ldots, P_m$ is a basis, then

$$\begin{aligned}
\bar{a}_{1s}P_1 + \bar{a}_{2s}P_2 + \cdots + \bar{a}_{ms}P_m &= P_s \\
\bar{a}_{1s}c_1 + \bar{a}_{2s}c_2 + \cdots + \bar{a}_{ms}c_m &= c_s - \bar{c}_s
\end{aligned}$$

where $\bar{a}_{is}$ and $\bar{c}_s$ are the coefficients of the corresponding canonical form.

43. Define linear spaces, vector spaces, dimensionality, affine vector geometry, a basis in a vector space, absolute coordinates, coordinates relative to a basis, convexity, convex hull, convex cone, rays, half-space, supporting half-spaces, hyper-planes. (Some of these terms are not defined in the text.)

44. Letting a vector $v \geq 0$ mean a vector of all nonnegative components, prove

(a) The equation $Ax = a$ has no solution $x \geq 0$ if and only if there exists a vector π such that $\pi A \leq 0$, $\pi a > 0$.

(b) The inequality system $Ax \leq a$ has no solution if and only if there exists a $\pi \geq 0$, such that $\pi A = 0$ and $\pi a < 0$.

(c) The inequality system $Ax \leq a$ has no solution $x \geq 0$ if and only if $\pi A \geq 0$ and $\pi a < 0$ for some π.

45. *Theorem:* Assume there are 4 sets of basic feasible solutions in a system $\sum_{j=1}^{m+2} P_j x_j = Q$, where P_j are m-component vectors.

$$
\begin{aligned}
(1)\quad & P_1a_1 + P_2a_2 + P_3a_3 + P_6a_6 + \cdots + P_{m+2}a_{m+2} = Q \\
& P_2b_2 + P_3b_3 + P_4b_4 + P_6b_6 + \cdots + P_{m+2}b_{m+2} = Q \\
& P_3c_3 + P_4c_4 + P_5c_5 + P_6c_6 + \cdots + P_{m+2}c_{m+2} = Q \\
& P_1d_1 + P_4d_4 + P_5d_5 + P_6d_6 + \cdots + P_{m+2}d_{m+2} = Q
\end{aligned}
$$

Then the basic solution

$$(2)\qquad P_1e_1 + P_2e_2 + P_5e_5 + P_6e_6 + \cdots + P_me_m = Q$$

is feasible if

$$(3)\qquad \frac{c_3 - a_3}{c_4} \leq \frac{b_3 - a_3}{b_4}$$

and

$$(4)\qquad \frac{a_3}{c_3} \leq \frac{a_k}{c_k} \qquad (k = 6, \ldots, m + 2)$$

and not feasible if (3) is false, or if (4) is false for some k and a selected range of values of b_k.

46. Let

$$\sum_{j=1}^{\infty} a_{ij}x_j = b_i \qquad (x_j \geq 0;\ i = 1, 2, \ldots, m)$$

be an infinite linear programming problem, which has a feasible solution. Prove that there is a feasible solution involving no more than m variables with $x_j > 0$.

47. *Theorem:* Let $(P_1, P_2, \ldots, P_m)$ be m linearly independent vectors in m-space and P_0 any other vector. If we let

$$x_1P_1 + x_2P_2 + \cdots + P_mx_m = P_0 + \begin{pmatrix} \varepsilon \\ \varepsilon^2 \\ \cdot \\ \cdot \\ \cdot \\ \varepsilon^m \end{pmatrix}$$

then there exists an ε_0 such that for all $0 < \varepsilon < \varepsilon_0$

$$x_i \neq 0 \qquad (i = 1, 2, \ldots, m)$$

48. (a) Consider a "Markov" system of equations

$$(1)\quad \begin{cases} (-1 + p_{11})x_1 + p_{12}x_2 + \cdots + p_{1n}x_n = 0 \\ p_{21}x_1 + (-1 + p_{22})x_2 + \cdots + p_{2n}x_n = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ p_{n1}x_1 + p_{n2}x_2 + \cdots + (-1 + p_{nn})x_n = 0 \end{cases}$$

$$(2)\quad x_1 + x_2 + \cdots + x_n = 1$$

where $p_{ij} > 0$ and $\sum_{i=1}^{n} p_{ij} = 1$ for $j = 1, 2, \ldots, n$. Prove that the first n equations in n unknowns are redundant; but if each equation i is modified by subtracting from it the last equation multiplied by $\lambda_i > 0$, where λ_i is chosen so that $p_{ij} - \lambda_i > 0$, then the corrected system of n-equations is non-redundant and there is a unique solution which is feasible, in fact, with $x_j > 0$.

(b) A system (1), where $\sum_{i=1}^{n} p_{ij} < 1$ for $j = 1, 2, \ldots, n$, and the constants (column of zeros) are replaced by $b_i < 0$, is referred to as a "Leontief" system. Show that such a system always has a unique feasible solution and that the above process can be used to reduce a Markov system to a Leontief system.

49. Prove or disprove for a three-equation system the conjecture that if x_1, x_2 are in the optimal basic set when the third equation is dropped, x_2, x_3 when the first equation is dropped, and x_1, x_3 when the second equation is dropped, then if x_1, x_2, x_3 forms a feasible basic set, it is optimal.

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